ELECTRICAL ENGINEERING TEXTS

# THE ELEMENTARY THEORY OF OPERATIONAL MATHEMATICS

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# ELECTRICAL ENGINEERING TEXTS

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ELECTRICAL ENGINEERING TEXTS

# THE ELEMENTARY THEORY OF OPERATIONAL MATHEMATICS

BY

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# PREFACE

The algebraic method as an aid to the simplification of work with the differential equations and integrals met with in the field of mathematical physics has been appreciated only partially in electrical engineering and hardly at all in any other field. Even in electrical engineering the lead of Oliver Heaviside has been inadequately followed up because of his mathematical limitations, which have seemed to narrow the horizon of those who have been interpreting his work.

This text is an outgrowth of an attempt (1) to search out the history of these methods; (2) to codify the set of theorems found; (3) to connect them with the work of the rigorists; and (4) to extend the theorems by all possible means. It is not well known that the history of these methods covers more than a century, and it is also not known what a wealth of work has already been done and written up and lost in the literature. A simple list of the theorems reads like a treatise on differential equations or a table of integrals. At present a number of mathematicians are doing some elegant work on the rigorous analysis of the operational forms and placing them in the category of high-grade mathematical tools, which mathematicians in general (let alone the practical workers) need not deign to use for their convenience. Moreover, there are many places in the theory where it is obvious that much immediate satisfactory research work can and needs to be done.

This text is not written for the mathematician, even though it will challenge his attention because of the fundamental character of the methods used. It is written primarily for all those who live and work to make mathematics useful to mankind, because of the inherent simplicity and beauty of the operational forms and because of their wide application to the daily tasks of the teacher and worker in our engineering schools and industrial laboratories.

Briefly, the essential nature of the method here presented is that of (1) "algebraizing"* the operators of the differential and

* This is the word used first by Oliver Heaviside in his characterization of his short methods which themselves come under the theory of this text.

integral calculus; (2) simplifying the algebraic forms thus obtained by the rules of ordinary algebra; and (3) then reinterpreting the resulting forms as operators, which are then found to be easier to use than the original ones. Great economy of time and space is thus gained, and difficult problems are resolved easily and quickly. The operations of the calculus in general, and the integration of differential equations in particular, are rendered as simple as are the solving of ordinary algebraic equations. An example or two will illustrate this.

To integrate the form $\int x^4 e^x dx$ we should use the formula of integration by parts four times with one further integration. By the operational method there is no integration, only some algebra and differentiation, *viz.:*

$\int x^4 e^{-x} dx \equiv \frac{1}{\mathrm{D}} e^{-x} x^4$ operational definition of integration

$\equiv e^{-x} \frac{1}{D-1} {}^{4}$ by operational shifting theorem

$\equiv -e^{x}(1 - D)^{-1} x^4$ algebraic reciprocal

$\equiv -e^{-x}(1 + D + D^2 + D^3 + D^4 + \cdots)x^4$ binomial theorem

$\equiv -e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24)$ simple differentiation

(No further powers of $D$ need be used, as they would produce zeros.)

The particular integral of the differential equation found in electrical engineering practice

$$(D^2 + 2Dh + h^2 + q^2)y = Ce^{-ht} \cos kt$$

is called its "steady-state solution."

Completing the square, we have

$$[(D + h)^2 + q^2]y = Ce^{-ht} \cos kt$$

Solve for $y$ algebraically;

$$y = \frac{1}{(D + h)^2 + q^2} \cdot Ce^{-ht} \cos kt$$

Use the "shifting theorem,"

$$y = Ce^{-ht} \frac{1}{D^2 + q^2} \cos kt$$

then the substitution of $-k^2$ for $D^2$, giving the result

$$y = Ce^{-ht} \frac{\cos kt}{k^2}$$

The text is full of illustrative examples of like simplification for the mathematician, the physicist, and the engineer.

Since the completion of this text, two books have been published to which special reference should be made—particularly so because no attempt is here made to justify rigorously any of the methods; only to show analytic parallelism. Full justification for all the methods will be found in the following:

H. T. Davis, "The Theory of Linear Operators," Principia Press, 1936.

E. G. Poole, "Introduction to the Theory of Linear Differential Equations," Clarendon Press, 1936.

For continued encouragement in the development of this work the author is indebted to Prof. Walter L. Upson, professor of electrical engineering, Washington University, St. Louis. Grateful acknowledgment is made to Dean Harry E. Clifford for his suggestions as to form; and to Dr. Francis Regan of St. Louis University for his rigorous criticism and reading of proofs.

Eugene Stephens.

St. Louis, Missouri,
*October*, 1937.

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# ELEMENTARY THEORY
## OF
# OPERATIONAL MATHEMATICS

## CHAPTER I

### INTRODUCTION AND DEFINITIONS

**§1. Introduction.**

(1) In the calculus and its applications, as in differential equations, a certain simplicity is obtained if we use single symbols for the forms for indicating differentiation and integration. In indicating differentiation, if, instead of using $\frac{d}{dx}$ we use simply $D$, we have substituted a single letter for three letters and a bar; and since integration is the inverse of differentiation, the suggestion is now obvious to use the algebraic inverse of $D$ to indicate integration; thus, $\frac{1}{D}$ or $D^{-1}$ for $\int ()dx$. We can then write $Df(x)$ or $D^{-1}f(x)$, respectively, for $\frac{d}{dx}f(x)$ or $\int f(x)dx$.

(2) In differential equations, powers of $D$ would be in order. Thus our differential equation for harmonic motion would become, $D^2y + n^2y = 0$, or, algebraically, $(D^2 + n^2)y = 0$. Here we have substituted a single form to operate on $y$ instead of the sum of two forms operating on $y$. Our simplification for the second derivative brought into the picture another simplification in the form of a single expression combining the whole story of the operations to be performed on $y$. Now, still another simplification is exposed (or rather suggested) by that form, *viz.*, the solving of the latter equation algebraically for $y$. Acting on the suggestion, we would have

$$y = \frac{1}{D^2 + n^2} \cdot 0 \qquad \text{or} \qquad = (D^2 + n^2)^-$$

If we try to interpret this form, we might say that the value of $y$ is an inverse operation upon zero. If this could only be performed, or if we could develop simple rules for such an operation, we might have revealed quickly the actual value of $y$ which would satisfy the algebraic equation and, what would be more to the point, the differential equation. The rules for handling the operator and forms containing it are really in existence, are easily used, and when handled with skill born of knowledge of their nature are powerful instruments for problems in the calculus. Two illustrations will suffice to indicate this.

(3) Let us integrate $x^2e^x$. Here the integral form is $\int x^2e^xdx$. The operational form is $D^{-1}x^2e^x$. A rule developed in Chap. II is "Shift $e^x$ across the operator by adding the coefficient of $x$ (in $e^x$) to the $D$"; *i.e.*,

$$D^{-1}e^xx^2 \equiv e^x(D + 1)^{-1}x^2$$

A second rule says "Expand the inverse operator into an infinite series in $D$"; *i.e.*:

$$(D + 1)^{-1} \equiv 1 - D + D^2 - D^3 + - \cdots$$

Substitute and operate by differentiation (instead of integration); thus,

$$\begin{aligned} e^x(D + 1)^{-1}x^2 &\equiv e^x(1 - D + D^2 - + \cdots)x^2 \\ &= e^x(x^2 - 2x + 2) \end{aligned}$$

We have

$$\int x^2e^xdx = e^x(x^2 - 2x + 2) + \text{constant of integration}$$

the indefinite integral. We have not integrated but have differentiated. Integration by parts twice will show that this is the correct value of the integral.

(4) The other example is the differential equation of motion $y'' + n^2y = 0$. Thus,

$$\begin{aligned} (D^2 + n^2)y &= 0 \\ y &= \frac{1}{D^2 + n^2}\,0 \end{aligned}$$

Turn $(D^2 + n^2)^{-1}$ into partial fractions; *i.e.*,

$$(D^2 + n^2)^{-1} \equiv (D + ni)^{-1}(D - ni)^{-1}$$
$$\equiv (2ni)^{-1}[(D - ni)^{-1} - (D + ni)^{-1}]$$

Then (ignoring the $1/2ni$ and the minus sign)

$$y = (D^2 + n^2)^{-1} \cdot 0 \equiv (D - ni)^{-1} \cdot 0 + (D + ni)^{-1} \cdot 0$$

Insert $e^{nix}e^{-nix} \equiv 1$ and $e^{-nix}e^{nix} \equiv 1$, respectively, between the operators and zero;

$$(D - ni)^{-1}e^{nix}e^{-nix} \cdot 0 \equiv e^{nix}(D - ni + ni)^{-1}(e^{-nix} \cdot 0)$$
$$\equiv e^{nix} \cdot D^{-1} \cdot 0$$
$$(D + ni)^{-1}e^{-nix}e^{nix} \cdot 0 \equiv e^{-nix}(D + ni - ni)^{-1}(e^{nix} \cdot 0)$$
$$\equiv e^{-nix} \cdot D^{-1} \cdot 0$$

Now interpret $D^{-1} \cdot 0$ as $\int 0dx = c$ and obtain

$$y = e^{nix}c_1 + e^{-nix}c_2$$

which is the complete solution of the differential equation, obtained operationally.

(5) The operational method is simple, easily acquired, and powerful, and we shall proceed to develop the formal rules forthwith.

**§2. The Necessity for Definitions.**

Since every subject must have a convenient notation, and every student of it a complete understanding of the terms and symbols used, our first concern is with the elementary terms which are used throughout this text. The following set are an irreducible minimum of fundamental ideas, in terms of which our subject is developed.

**§3. Definitions.**

(1) *Subject.* The subject is the function upon which an operation is to be carried out.

(2) *Operator.* An operator is a symbol that is used in place of a description of a mathematical process or procedure. It symbolizes the specific instruction or set of instructions for process or procedure. Operators may be monomials, polynomials, linear, compound, iterative, or functional.

*a*. Monomials (single symbols):

$$b \cdot \mathbf{a} = c$$
$$\psi \cdot x^m = (x + h)^r$$
$$\Delta \cdot a^x = a^{x+h}$$
$$D \cdot u \equiv \frac{d}{dx} u \equiv \underset{h\to 0}{\mathbf{L}} \frac{u_{x+h} - u_x}{h} \equiv \underset{h\to 0}{\mathbf{L}} \frac{\Delta}{h} \cdot u$$
$$d_1 \cdot f(x, y) \equiv \frac{\partial}{\partial x} f(x, y) \equiv \underset{h\to 0}{\mathbf{L}} \frac{f(x + h, y_0) - f(x, y_0)}{h}$$
$$\vartheta \cdot u \equiv x \cdot D \cdot u \equiv x \cdot \frac{d}{dx} u$$

These are defining equations. Operators are known when the results of their action are known.

*b*. Polynomials:

$$\Delta \equiv \psi - 1, \qquad \psi \equiv \Delta + 1$$
$$F(D) \equiv \sum_{k=n}^{0} A_k D^k, \qquad e^{hD} \equiv \sum_{k=0}^{\infty} \frac{1}{k!} (hD)^k$$
$$\nabla \equiv \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \equiv d_1 + d_2 + d_3$$
$$\pi \equiv \sum_i a_i d_i \equiv a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} +$$

In the foregoing it is seen that the process of symbolizing can be a cumulative one. The left-hand symbol is one covering several separate and simultaneous operations indicated on the right-hand side. When this is the case, they are said to be identical, and the identity sign is used.

*c*. Linear operators are those in which the operative symbols appear to the first degree only.

*d*. A compound operator consists of a group of simple, defined, monomial, or polynomial operators, the subject of each of them after the first being the result of all the preceding operations. Thus:

$$\Delta \cdot \psi \cdot a \cdot x^2 = a(2hx + 3h^2)$$

for

$$\begin{aligned} a(x^2) &= ax^2 \\ \psi(ax^2) &= a(x + h)^2 \\ \Delta[a(x + h)^2] &= a(x + 2h)^2 - a(x + h)^2 \\ &= a(2hx + 3h^2) \end{aligned}$$

*e.* Repeated (iterated) operators are indicated by an exponent (index). Thus:

$$\begin{aligned} D^3 \cdot x^4 &= D \cdot D \cdot D \cdot (x^4) \\ &= D \cdot D \cdot (4x^3) \\ &= D \cdot (12x^2) \\ &= 24x \end{aligned}$$

*f.* A functional operator is one that is algebraic or transcendental in form but not simple or compound. It includes rational or irrational fractions of simple operators, or trigonometric or logarithmic forms. They all must be capable of expansion by the known methods of algebra or calculus into ascending or descending powers of the simple operator involved. Examples would be:

$$\begin{aligned} \frac{1}{D} &\equiv 1 + D + D^2 + \\ e^{hD} &\equiv 1 + hD + \frac{1}{2!}h^2D^2 + \cdots \\ \sin D &\equiv D - \frac{1}{3!}D^3 + \frac{1}{5!}D^5 - \cdots \\ \log(1 + D) &\equiv D - \tfrac{1}{2}D^2 + \tfrac{1}{3}D^3 - \cdots \\ \sqrt{1 - d^2} &\equiv 1 - \tfrac{1}{2}d^2 + \tfrac{1}{8}d^4 - \cdots \end{aligned}$$

(3) *Order.* Unless the order of performance of the single operations indicated by a group or compound operator is indifferent because of mutual relations, the operators should be ordered; *i.e.*, the operations should be performed in the same succession as the symbols are written, from right to left. If the order is thus relevant, the separate operators are said to be *noncommutative;* if indifferent, commutative, or permutable. [*cf.* Commutation, (6).] Thus:

$D$ and $x$ are noncommutative;

$$\begin{aligned} D \cdot x \cdot (x^2) &= Dx^3 = 3x^2 \\ x \cdot D \cdot (x^2) &= x(2x) = 2x^2 \end{aligned}$$

$D$ and $a$ are commutative;

$$\begin{aligned} D \cdot a \cdot (x^2) &= D \cdot ax^2 = 2ax \\ a \cdot D \cdot (x^2) &= a \cdot 2x = 2ax \end{aligned}$$

(4) *Result.* That function or quantity which is obtained by carrying out the operation upon the subject, connected with the operator subject by an equality sign. Thus:

$$\psi \cdot x^2 = (x + h)^2$$
$$(\text{Operator}) \cdot (\text{subject}) = (\text{result})$$

(5) *Elementary Algebraic Transformations.* *a.* Indicated operations of addition, subtraction, multiplication, division, evolution, involution, and transcendentals.

*b.* Identical substitutions.

*c.* Commutation.

*d.* Expansion into series.

(6) *Commutation.* Transmutation in place, or permutation in order of the component consecutive operators of a group or compound operator.

(7) *Interpretation.* An operator is interpretable if it can be transformed by elementary algebraic transformations into a series of simple or elementary known operators, each of which can be carried out in order. Thus:

$$e^{hD} \cdot S \equiv \left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \quad \cdot \right) \cdot S$$
$$S + hS' + \frac{h^2}{2!}S'' + \cdot \cdot \cdot$$

which makes it possible to interpret $e^{hD}$ from the result.

(8) *Inverse.* In general, an inverse operation is that which when performed upon a result will bring back the original subject unchanged. Thus:

$$\frac{1}{a}[a(x^2)] = \frac{1}{a}(ax^2) = x^2$$
$$Q^{-1} \cdot Q \cdot u = u$$

The inverse operator uses the same symbol as the direct but with $-1$ as its exponent.

An inverse of a compound operator will be defined as the inverse of its component parts in the reverse order. This definition is easily seen to be a logical one, if we consider that in the second example above $Q^{-1} \cdot Q \equiv 1$ if the relation is to be true. Using this, we can see that

if

$$Q \equiv p \cdot q \cdot r$$

then

$$\begin{aligned}
Q^{-1} \cdot Q &\equiv Q^{-1} \cdot p \cdot q \cdot r \\
1 &\equiv Q^{-1} \cdot p \cdot q \cdot r \\
r^{-1} &\equiv Q^{-1} \cdot p \cdot q \cdot (r \cdot r^{-1}) && Q^{-1} \cdot p \cdot q \\
q^{-1} &\equiv Q^{-1} \cdot p \cdot (q \cdot q^{-1}) && \equiv Q^{-1} \cdot p \\
r^{-1} \cdot q^{-1} \cdot p^{-1} &\equiv Q^{-1} \cdot (p \cdot p^{-1}) && \equiv Q^{-1}
\end{aligned}$$

or

$$Q^{-1} \equiv r^{-1} \cdot q^{-1} \cdot p^{-1}$$

(9) *Equivalency.* *a.* Two operators may be used equivalently if one can be obtained from the other by the use of elementary algebraic transformations.

*b.* Provided, that one can be interpreted. "The mere processes of symbological reasoning are independent of the conditions of their interpretation." [Boole, "Differential Equations," 2d ed., p. 399; *i.e.*, an experimental principle, not a mathematical theorem.]

*c.* But the results of equivalent operations upon the same function are not necessarily identical.

This can be seen to be reasonable from the fact of the indeterminacy of the indefinite integral in the calculus.

(10) *Validity.* The validity of the result of an operation may be established by the use of the inverse, *i.e.*, as in the direct and inverse operations of the calculus.

(11) *Abstraction.* When relations exist between operators independently of the form or value of the subject, as under the principle of equivalency, we may *abstract* the consideration of the operators themselves, ignoring the subject; and then an equivalency sign [$\equiv$] between the operators is understood to indicate that they are universally equivalent.

This is important from the point of view of pure operational mathematics. As in arithmetic, we abstract number relations from the counting and combination of concrete things, so in operational work we can abstract operational relations from the subjects and consider the algebraic or analytical relations obtaining between the operators themselves. The whole method

consists essentially of this type of abstraction. The following quotations from the article "Operation" in the *Penny Encyclopædia*, published nearly a century ago, would not be inapropos:

The use of symbols of operation not standing for magnitude but for directions how to proceed with magnitudes began with Leibniz and Newton, before whose time algebraical characters denoted simple numbers. The progress of the differential calculus forced the attention of mathematicians upon modes of denoting, not results of processes, but ways of proceeding. The generalizations arising out of the organization which notation gave to processes led to the use of indefinite and arbitrary symbols of operation. Finally it was observed that the symbols of operation employed in many general theorems would give simple and well-known relations if their meaning as symbols of operation were forgotten and they were considered as symbols of quantity.

. . . . .

In this way many theorems were investigated, upon the following plan of proceeding:—1. Throw away symbols of quantity from a known theorem, and proceed in any manner which may appear eligible with the symbols of operation, treating them as if they were quantities. 2. When a result has been obtained, restore the symbols of quantity to their old places, and those of operation to their old meaning. 3. The result as thus viewed has all the presumption in its favor which arises from its being the legitimate consequence of a method which, whether accurate or not, has never been found to lead to anything but what could otherwise be satisfactorily shown to be true. And though Lagrange himself, Arbogast, the English translators of Lacroix, Brinkley, etc., may have used language in reference to this method which would seem to imply that they considered it as one of demonstration, yet it is obvious, from the care taken by them to have abundant external verification in every case, that their results were considered by themselves as resting on such a presumption as that above noted; though it is also evident that they considered the presumption as amounting to moral certainty, which indeed they were justified in doing.

In reality, one can easily see, from the preceding, and the definitions of this section, that this subject is

. . . neither a resemblance of an algebra, nor a calculus of functions founded on algebra, but **an algebra**, or if the reader pleases algebra itself; so that its conclusions rest upon the same foundations as those of ordinary algebra.

# CHAPTER II

## THE OPERATOR $D \equiv \frac{d}{dx}$

### §4. Definitions and Elementary Algebra.

(1) *Definition.* The symbol $D$ has been fairly universally adopted as that for differentiation. But its use as an algebraic symbol in the simplification of applications to the calculus has been singularly frowned upon in certain mathematical circles. We are here merely defining it by the form

$$D \equiv \frac{d}{dx} \tag{1}$$

and its inverse by

$$\begin{aligned} D^{-1} \equiv \frac{1}{D} &\equiv \int^{x} ()dx + c \\ &\equiv \int_{a}^{x} ()dx \end{aligned} \tag{2}$$

Reference should be made to the definition of an "inverse" [see §3 (8)] and to "inverse operations and the appendage" [Appendix III, §60 (6), (7), (8)] for a proper understanding of the use of the constant of integration in operational work; and its full significance will transpire in the present chapter (§7) under "Operations on Zero."

(2) When a single independent variable, other than $x$, is used, the symbol $D$ may still be used, though sometimes (particularly in electrical usage) $p$ is found as

$$p \equiv \frac{d}{dt}$$

where time $t$ is the independent variable. This is the symbol used by Oliver Heaviside and by Paul Levy.* But it must be

* Le calcul symbolique d'Heaviside, *Bull. sci. math.*, (2) **50** (1926), 174–192. Levy, however, uses $D$ in his paper, Sur la dérivation et l'intégration généralisées, same publication, pp. 307–352.

remarked that the symbol for differentiation and for integration can be any letter or character that will be so defined. We must not be slaves to symbols in this operational work. It is relationships and their forms that are essential.

(3) *Iteration.* With $D \equiv \frac{d}{dx}$, repeated or iterated differentiation can be indicated by the exponential form

$$\begin{aligned} D^n &\equiv D \cdot D \cdot \cdot \cdot D \\ &\equiv \frac{d}{dx} \cdot \frac{d}{dx} \cdot \cdot \cdot \frac{d}{dx} \equiv \frac{d^n}{dx^n} \end{aligned} \tag{3}$$

Thus $D^n \cdot f(x)$ calls for the $n$ successive differentiations of $f(x)$.

(4) Likewise, iterated integration is indicated by

$$\begin{aligned} D^{-n} &\equiv D^{-1} \cdot D^{-1} \cdot D^{-1} \cdot \cdot \cdot D^{-1} \\ &\equiv \frac{1}{D} \cdot \frac{1}{D} \cdot \frac{1}{D} \cdot \cdot \cdot \frac{1}{D} \\ &\equiv \int dx \cdot \int dx \cdot \int dx \cdot \cdot \cdot \int dx() \\ &\equiv \int_a^x \cdot \cdot \cdot \int_a^x ()dx^n \end{aligned}$$

or

$$\equiv \int^x \cdot \cdot \cdot \int^x ()dx^n + \sum_{i=0}^{n-1} C_i x^i \tag{4}$$

(5) *Distribution.* $D$, any power of $D$, and therefore any rational integral function of $D$ are distributive operators; *i.e.*, they can be performed upon any sum of subjects separately. Thus

$$\begin{aligned} D \cdot (u + v) &\equiv D \cdot u + D \cdot v \\ D \cdot [f(x) + g(x) + \cdot \cdot \cdot ] &\equiv D \cdot f(x) + D \cdot g(x) + \cdot \cdot \cdot \\ D^n \cdot (u + v + \cdot \cdot \cdot ) &\equiv D^n \cdot u + D^n \cdot v + \cdot \cdot \cdot \\ D^n \cdot [f(x) + g(x) + \cdot \cdot \cdot ] &\equiv D^n \cdot f(x) + D^n \cdot g(x) + \cdot \cdot \cdot \\ F(D) \cdot [u + v + \cdot \cdot \cdot ] &\equiv F(D) \cdot u + F(D) \cdot v + \cdot \cdot \cdot \end{aligned}$$

(6) By $F(D)$ we mean a rational integral function of $D$ of the type

$$F(D) \equiv \sum_{k=0}^{n} a_k D^k \equiv a_0 + a_1 D + a_2 D^2 + \cdot \cdot \cdot + a_n D^n \tag{5}$$

the $a_k$ being fixed constants in the complex field.*

* PINCHERLE and AMALDI, "Le operazioni distributive," Bologna, 1902.

(7) $F(D)$ can also include any operator in $D$ that can be expanded into such a form by Taylor's theorem or by actual algebraic operations; thus:

$$e^D \equiv 1 + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots$$

$$\frac{1}{1 + D} \equiv 1 - D + D^2 - D^3 +$$

$$\sin D \equiv D - \frac{1}{3!}D^3 + \frac{1}{5!}D^5 - \frac{1}{7!}D^7 + \cdots$$

$$\log (1 + D) \equiv D - \frac{1}{2}D^2 + \frac{1}{3}D^3 - \frac{1}{4}D^4 + \cdots$$

(8) *Commutation.* Differential operators are commutative with respect to both addition and multiplication. With $D \equiv \frac{d}{dx}$, $p \equiv \frac{d}{dt}$, and $c$ a fixed constant, we have

$$D + p \equiv p + D \qquad \text{for} \qquad \frac{du}{dx} + \frac{du}{dt} \equiv \frac{du}{dt} + \frac{du}{dx} \tag{6}$$

$$D \cdot p \equiv p \cdot D \qquad \text{for} \qquad \frac{d^2u}{dx \cdot dt} \equiv \frac{d^2u}{dt \cdot dx} \tag{7}$$

$$D \cdot c \equiv c \cdot D \qquad \text{for} \qquad \frac{dcu}{dx} \equiv c\frac{du}{dx} \tag{8}$$

From this last equation (8) it follows that any rational integral function of $D$ or any function expansible into such is commutative with any other of the same kind; *i.e.*,

$$F(D) \cdot f(D) \cdots \equiv f(D) \cdot F(D) \cdots \tag{9}$$

A particular case of this would be

$$(D - a) \cdot (D - b) \cdots \equiv (D - b) \cdot (D - a) \cdots$$

where $a$ and $b$ are fixed constants.

(9) *Index Law.* For positive integral powers of $D$ we shall have

$$D^m \cdot D^n \equiv D^{m+n} \tag{10}$$

and, in general,

$$[F(D)]^m \cdot [F(D)]^n \equiv [F(D)]^{m+n} \tag{11}$$

(10) For negative values of the exponents, if we neglect the appendage, we can state the following:

$$D^m \cdot D^{-n} \equiv D^{-n} \cdot D^m \equiv D^{m-n} \tag{12}$$

from which, for $m \equiv n$,

$$D^m \cdot D^{-m} \equiv D^{-m} \cdot D^m \equiv D^0 \equiv 1 \tag{13}$$

Equation (13) is in line with our general definition of an inverse [see §3 (8)]. Also, in general,

$$[F(D)]^m \cdot [F(D)]^{-m} \equiv [F(D)]^0 \equiv 1 \tag{14}$$

(11) Attention is called to the effect of consideration of the appendage upon the principle of commutation. We have

$$\begin{aligned} D \cdot D^{-1} \cdot f(x) &\equiv \frac{d}{dx}\int f(x)dx \\ &\equiv \frac{d}{dx}[F(x) + C] \\ &\equiv \frac{d}{dx}F(x) + \frac{dC}{dx} \\ &\equiv f(x) \end{aligned}$$

But

$$\begin{aligned} D^{-1} \cdot D \cdot f(x) &\equiv \int\left[\frac{d}{dx}f(x)\right]dx \\ &\equiv f(x) + K \end{aligned}$$

Thus

$$D \cdot D^{-1} \equiv 1, \qquad \text{but} \qquad D^{-1} \cdot D \not\equiv 1.$$

But if we think of $f(x)$ as an entire function of $x$ and as being restored intact by the inverse of either operation $D$ or operation $D^{-1}$, then $D \cdot D^{-1} \equiv D^{-1} \cdot D \equiv 1$. We can even think of $f(x)$ as a particular function and $D^{-1} \cdot D$ as introducing the appendage. Then with the proper conditions on the result the arbitrary constant could be evaluated in such a way as to bring the result to the original function. In this way, the final results would be the same as if the two operators were commutative. This could also be done with $F(D)$, in general, if we so desired. And in this text we are going to work in just that way unless otherwise specified. We shall see in the sequel that this is just as well.

(12) *Expansion of Inverse Operators.* When we wish to use the inverse of a rational integral function of $D$, we frequently change its form into a series of ascending or descending powers of $D$. Sometimes there are several series that can be obtained, and we have to choose between them.

(13) Division is the usual and simplest way of obtaining the series, and this can be done in more than one way. For instance, with the operator $D - a$, the inverse $\dfrac{1}{D-a}$ can be expanded by division:

$$
\begin{array}{ll}
1 & \dfrac{1}{D} \\
\underline{1 - \dfrac{1}{D}a} & \\
\dfrac{1}{D}a & +\dfrac{1}{D^2}a \\
\dfrac{1}{D}a - \dfrac{1}{D^2}a^2 & \\
\overline{D} & +\dfrac{1}{D^3}a^2 \\
 & + \cdots
\end{array}
$$

so that we have

$$
\frac{1}{D-a} \equiv \sum_{k=1}^{\infty} a^{k-1} \cdot D^{-k} \tag{15}
$$

If we divide as follows:

$$
\begin{array}{l|ll}
\underline{-a + D} & 1 & -\dfrac{1}{a} \\
 & \underline{1 - \dfrac{1}{a}D} & \\
 & \dfrac{1}{a}D & -\dfrac{1}{a^2}D \\
 & \underline{\dfrac{1}{a}D - \dfrac{1}{a^2}D^2} & \\
 & \dfrac{1}{a^2}D^2 & -\dfrac{1}{a^3}D^2
\end{array}
$$

we obtain

$$
\frac{1}{D-a} \equiv -\sum_{k=0}^{\infty} a^{-k-1} \cdot D^{k} \tag{16}
$$

Equations (15) and (16) are essentially different with respect to their action as operators. If (15) is used on a rational integral function of $x$, we shall have an infinite series. But if (16) is

used on the same function, we have an entire expression as a result. For example,

$$\frac{1}{D-a}\cdot x^2 \equiv \left[\frac{1}{D} + a\frac{1}{D^2} + a^2\frac{1}{D^3} + \cdots\right]\cdot x^2$$
$$= \frac{1}{3}x^3 + \frac{a}{3\cdot 4}x^4 + \frac{a^2}{3\cdot 4\cdot 5}x^5 + \cdots$$
$$\frac{1}{D-a}\cdot x^2 \equiv \left[-\frac{1}{a} - \frac{1}{a^2}D - \frac{1}{a^3}D^2 - \cdots\right]\cdot x^2$$
$$= -\frac{1}{a}x^2 - \frac{2}{a^2}x - \frac{2}{a^3}$$

The question would then arise as to which of these results is the valid one. The answer would depend entirely upon the connection that the operation has upon other considerations. The values of $x$ that make either of these results finite would have to be determined. We should have to exclude values that would make either infinite. It may even be possible that the sum of both of these would under certain conditions be all right. However, attention must be called to the fact that since the identities (15) and (16) are identities of operators only, no question need be raised until after the operations are performed and the results obtained.

(14) If we have an inverse function like $\dfrac{1}{(D-a)(D-b)}$, then we shall have three forms of expansion to choose from, one of which is called a Laurent series. We should thus have three different results after operation upon a given subject. These three would have to be examined carefully to see which would satisfy the conditions under which our result is desired to be valid.

(15) Expansions can always be safely made if we use Taylor's theorem. For simple operators like $(D - a)^{-1}$, expansion by the binomial theorem produces the same result as by Taylor's theorem.

(16) As exercises, the student should expand the following, each in as many ways as possible:

(*a*) $(D - 1)^{-1}$.

(*b*) $(D - 1)^{-2}$.

(*c*) $[(D - 1)(D - 2)]^{-1}$.

(*d*) $(D^2 - 1)^{-1}$.

(*e*) $(D^2 + 1)^{-1}$.

Also, by means of the expansions of the operators, obtain the results and compare them for the following:

(*f*) $(D - 1)^{-1} \cdot x^4$.

(*g*) $(D - 1)^{-2} \cdot x$.

(17) *Partial Fractions.* It is often necessary in operational work to expand inverse operators into partial fractions. The algebra of partial fractions is given here. Though a chapter on partial fractions can be found in all college algebras, most of them do not give the simplest methods, which are by the use of the calculus. These methods are detailed in the following paragraphs.

(18) Consider the rational fractional function $\frac{f(D)}{\phi(D)}$ with the degree of $f(D)$ lower than that of $\phi(D)$. By the theory of equations $\phi(D)$ can be separated into factors of the following types only:

Type I: $D - a$.

Type II: $(D - a)^p$.

Type III: $[(D - a)^2 + b^2]$.

Type IV: $[(D - a)^2 + b^2]^p$.

(19) *Types* I and II will be treated by the method given by Joseph Edwards.*

(20) *Type* I: $(D - a)$ *Present Once.* Here we have

$$\phi(D) \equiv (D - a)\alpha(D),$$

which will give us

$$\frac{f(D)}{\phi(D)} \equiv \frac{f(D)}{(D - a)\alpha(D)} \equiv \frac{\quad}{D - a} + \frac{g(D)}{\alpha(D)} \tag{17}$$

The numerator $A$ is to be determined as a constant. We shall have

$$\frac{f(D)}{\alpha(D)} \equiv A + \frac{g(D)(D - a)}{\alpha(D)}$$

with $D \equiv a$

$$\frac{f(a)}{\alpha(a)} \equiv A$$

* "Integral Calculus," vol. I, §139, pp. 143*ff*.

and we then obtain

$$\frac{f(D)}{\phi(D)} \equiv \frac{f(a)}{\alpha(a)} \cdot \frac{1}{D - a} + \frac{g(D)}{\alpha(D)}$$

The first term on the right of the identity sign is the partial fraction corresponding to the factor $(D - a)$ in $\phi(D)$.

(21) If all factors of $\phi(D)$ are distinct and of Type I, the method can be repeated for each one. Thus; in the example,

$$\frac{}{(D-1)(D-2)(D-5)} \quad \frac{}{D-1} + \frac{B}{D-2} + \frac{C}{D-5}$$

$$\left.\frac{1}{(D-2)(D-5)}\right|_{D=1} \quad \frac{1}{(1-2)(1-5)} = \frac{1}{4} = A$$

$$\left.\frac{1}{(D-1)(D-5)}\right|_{D=2} \quad \frac{1}{(2-1)(2-5)} \quad -\frac{1}{3} = B$$

$$\left.\frac{1}{(D-1)(D-2)}\right|_{D=5} \quad \frac{1}{(5-1)(5-2)} = \frac{1}{12} = C$$

and

$$\frac{1}{(D-1)(D-2)(D-5)} \equiv \frac{1}{4(D-1)} - \frac{1}{3(D-2)} + \frac{1}{12(D-5)}$$

(22) *Type* II: $(D - a)$ *Repeated* $p$ *Times.* Here

$$\phi(D) \equiv (D - a)^p \cdot \alpha(D),$$

which will give us

$$\frac{f(D)}{\phi(D)} \equiv \frac{f(D)}{(D-a)^p \cdot \alpha(D)} = \frac{A_1}{D-a} + \frac{A_2}{(D-a)^2} + \frac{A_3}{(D-a)^3} + \cdots + \frac{A_p}{(D-a)^p} + \frac{g(D)}{\alpha(D)} \quad (18)$$

Now put $D - a = y$, so that $D = y + a = a + y$. Then

$$\left.\frac{f(D)}{(D-a)^p \cdot \alpha(D)}\right._{D=a+y} = \frac{1}{y^p} \cdot \left[\frac{f(a+y)}{\alpha(a+y)}\right]$$

By division of $f(a + y)$ by $\alpha(a + y)$ we obtain an ascending power series in $y$, the coefficients of the first $p$ terms of which will be, respectively, $A_p$, $A_{p-1}$, . . . $A_2$, $A_1$, as can be shown as follows:

$$\frac{1}{y^p}\left[\frac{f(a+y)}{\alpha(a+y)}\right] \equiv \frac{1}{y^p}\left[A_p + A_{p-1}y + A_{p-2}y^2 + \cdots + A_1 y^{p-1} + \frac{h(y)y^p}{\alpha(a+y)}\right]$$

$$\equiv \frac{A_p}{y^p} + \frac{A_{p-1}}{y^{p-1}} + \frac{A_{p-2}}{y^{p-2}} + \cdots + \frac{A_1}{y} + \frac{h(y)}{\alpha(a+y)}$$

Now, changing back $y = D - a$, we have

$$\frac{1}{y^p}\left[\frac{f(a+y)}{\alpha(a+y)}\right]_{y=D-a}$$

$$\equiv \frac{A_p}{(D-a)^p} + \frac{A_{p-1}}{(D-a)^{p-1}} + \cdots + \frac{A_1}{D-a} + \cdots$$

Each repeated factor in $\phi(D)$ will be treated separately in the preceding manner for the respective coefficients due to that factor.

(23) If $\alpha(D)$ is a single linear factor, the division can be performed by a sort of synthetic division process which gives the coefficients quickly. An example of this would be

$$\frac{1}{(D-1)^3(D+1)}\Bigg|_{D-1=y} = \frac{1}{y^3}\left[\frac{1}{2+y}\right]$$

Dividing

$$\begin{array}{l|llll} 2 & 1 & & & \\ & \tfrac{1}{2} & \tfrac{1}{4} & -\tfrac{1}{8} & \tfrac{1}{16} \\ \hline & \tfrac{1}{2} & -\tfrac{1}{4} & \tfrac{1}{8} & -\tfrac{1}{16} \end{array}$$

we have

$$\frac{1}{2+y} \equiv \frac{1}{2} - \frac{1}{4}y + \frac{1}{8}y^2 - \frac{1}{16}\frac{y^3}{(2+y)}$$

so that

$$\frac{1}{y^3}\left[\frac{1}{2+y}\right] \equiv \frac{1}{2y^3} - \frac{1}{4y^2} + \frac{1}{8y} - \frac{1}{16(2+y)}$$

Substituting back, we have

$$\frac{1}{(D-1)^3(D+1)}$$

$$\equiv \frac{1}{2(D-1)^3} - \frac{1}{4(D-1)^2} + \frac{1}{8(D-1)} - \frac{1}{16(D+1)}$$

(24) An example of two repeated factors is the following:

$$\frac{1}{(D-1)^3(D+1)^2} \equiv \frac{A}{(D-1)^3} + \frac{B}{(D-1)^2} + \frac{C}{D-1} + \frac{E}{(D+1)^2} + \frac{F}{D+1}$$

Since when $D - 1 = y$, $D + 1 = 2 + y$, and

$$(D+1)^2 = 4 + 4y + y^2$$

we have

$$\left.\frac{1}{(D-1)^3(D+1)^2}\right|_{D-1=y} = \frac{1}{y^3}\left[\frac{1}{4+4y+y^2}\right]$$

Dividing

$$\begin{array}{l|l|l} \underline{4 \quad 4 \quad 1} & 1 & \tfrac{1}{4} = A \\ & \underline{1 + 1 + \tfrac{1}{4}} & \\ & \quad -1 - \tfrac{1}{4} & -\tfrac{1}{4} = B \\ & \underline{\quad -1 - 1 \quad - \tfrac{1}{4}} & \\ & \qquad \tfrac{3}{4} + \tfrac{1}{4} & \tfrac{3}{16} = C \end{array}$$

When $D + 1 = y$, $D - 1 = -2 + y$, and

$$(D-1)^3 = -8 + 12y - 6y^2 + y^3$$

and

$$\left.\frac{1}{(D-1)^3(D+1)^2}\right|_{D+1=y} = \frac{1}{y^2}\left[\frac{1}{-8+12y-6y^2+y^3}\right]$$

Dividing

$$\begin{array}{l|l|l} \underline{-8 \quad +12 \quad -6 \quad +1} & 1 & -\tfrac{1}{8} = E \\ & \underline{1 - \tfrac{3}{2} + \tfrac{3}{4} - \tfrac{1}{8}} & \\ & \quad \tfrac{3}{2} - \tfrac{3}{4} + \tfrac{1}{8} & -\tfrac{3}{16} = F \end{array}$$

we thus have

$$\frac{1}{(D-1)^3(D+1)^2} \equiv \frac{1}{4(D-1)^3} - \frac{1}{4(D-1)^2} + \frac{3}{16(D-1)} - \frac{1}{8(D+1)^2} - \frac{3}{16(D+1)}$$

(25) *Types* III and IV are treated by the method given by Williamson.*

* "Integral Calculus," §41, pp. 49*ff*.

(26) *Type* III: *One Pair of Imaginary Roots* $a \pm ib$. Here $\phi(D) \equiv [(D-a)^2 + b^2]\cdot\alpha(D)$, or $\phi(D) \equiv [D^2 + pD + q]\cdot\alpha(D)$, where $p = -2a$ and $q = a^2 + b^2$. Then

$$\frac{f(D)}{\phi(D)} = \frac{f(D)}{(D^2+pD+q)\cdot\alpha(D)} = \frac{LD+M}{D^2+pD+q} + \frac{h(D)}{\alpha(D)} \qquad (19)$$

The partial fraction corresponding to the factor $D^2 + pD + q$ is the first term on the right of the identity sign. Then we have

$$\frac{f(D)}{\alpha(D)} \equiv LD + M + \frac{h(D)}{\alpha(D)}(D^2+pD+q) \qquad (20)$$

Now set $D^2 + pD + q = 0$, or $D^2 = -pD - q$, obtaining

$$f(D) \equiv \alpha(D)(LD+M)$$

in which every $D^2$ must be set equal to $-pD - q$. When this equation is simplified, and that is done, we shall have an identity in $D$ and can equate coefficients, which will give us two linear equations in $L$ and $M$, sufficient for their determination.

(27) An example of this is

$$\frac{2D-1}{(D^2+2D+5)(D-1)(D-2)} \equiv \frac{LD+M}{D^2+2D+5} + \frac{A}{D-1} + \frac{B}{D-2}$$

Multiply through by $D^2 + 2D + 5$:

$$\frac{2D-1}{(D-1)(D-2)} \equiv LD + M + \frac{h(D)}{(D-1)(D-2)}(D^2+2D+5)$$

Set $D^2 + 2D + 5 = 0$, and multiply through by $(D-1)(D-2)$:

$$\begin{aligned} 2D - 1 &\equiv (D-1)(D-2)(LD+M) \\ &\equiv LD^3 - 3LD^2 + MD^2 + 2LD - 3MD + 2M \end{aligned}$$

Set $D^2 = -2D - 5$, and collect terms on the right:

$$2D - 1 \equiv -2LD^2 + (-5L - 2M + 2L - 3M)D - 5M + 2M$$

Again set $D^2 = -2D - 5$ and collect terms:

$$2D - 1 \equiv (L - 5M)D + (10L - 3M)$$

Now equate coefficients:

$$\begin{aligned} L - 5M &= 2 \\ 10L - 3M &= -1 \end{aligned}$$

from which

$$L = -11/47, \qquad M = -21/47$$

Thus the partial fraction due to $D^2 + 2D + 5$ is found to be

$$\frac{-11D - 21}{47(D^2 + 2D + 5)}$$

(28) *Type* IV: *A Pair of Imaginary Roots Repeated* k *Times.* Here $\phi(D) \equiv (D^2 + pD + q)^k \cdot \alpha(D)$, and

$$\frac{f(D)}{\phi(D)} \equiv \frac{f(D)}{(D^2 + pD + q)^k \cdot \alpha(D)}$$
$$\equiv \frac{L_kD + M_k}{(D^2 + pD + q)^k} + \frac{L_{k-1}D + M_{k-1}}{(D^2 + pD + q)^{k-1}} + \cdots + \frac{L_1D + M_1}{D^2 + pD + q} + \frac{h(D)}{\alpha(D)} \tag{21}$$

There are $k$ partial fractions corresponding to the factor

$$(D^2 + pD + q)^k$$

Multiply through by that factor, obtaining

$$\frac{f(D)}{\alpha(D)} \equiv L_kD + M_k + (L_{k-1}D + M_{k-1})(D^2 + pD + q) + \cdots + \frac{h(D)}{\alpha(D)}(D^2 + pD + q)^k \tag{22}$$

Set $D^2 = -pD - q$

$$\frac{f(D)}{\alpha(D)} \equiv L_kD + M_k \tag{23}$$

This equation is now treated exactly as in Type III, giving the values of $L_k$ and $M_k$. For the next set $L_{k-1}$ and $M_{k-1}$, differentiate equation (22) with respect to $D$, obtaining

$$\frac{d}{dD}\left[\frac{f(D)}{\alpha(D)}\right] \equiv L_k + [L_{k-1}D + M_{k-1}](2D + p) + (D^2 + pD + q)(L_{k-1} + K) \tag{24}$$

where $K$ is immaterial, since we are setting $D^2 = -pD - q$;

$$\frac{d}{dD}\left[\frac{f(D)}{\alpha(D)}\right] \equiv L_k + [L_{k-1}D + M_{k-1}](2D + p) \tag{25}$$

When this equation is simplified, and all $D^2$ are finally set equal to $-pD - q$, we may equate coefficients and obtain two linear equations in $L_{k-1}$ and $M_{k-1}$. For the third set of coefficients $L_{k-2}$ and $M_{k-2}$, differentiate equation (22) twice with respect to $D$ and proceed as for the second set. Repetitions of this method eventually produce all $k$ sets of coefficients.

(29) A simple example of Type IV is

$$\frac{D^3 + D}{(D^2 + D + 1)^2} - \frac{L_2D + M_2}{(D^2 + D + 1)^2} + \frac{L_1D + M_1}{D^2 + D + 1}$$

Multiplying through by $(D^2 + D + 1)^2$, we have

$$D^3 + D \equiv L_2D + M_2 + (L_1D + M_1)(D^2 + D + 1) \qquad (a)$$

Set $D^2 = -D - 1$

$$D(-D - 1) + D \equiv L_2D + M_2$$

or

$$-D^2 - D + D \equiv L_2D + M_2$$

and

$$-(-D - 1) \equiv L_2D + M_2$$
$$D + 1 \equiv L_2D + M_2$$

from which

$$L_2 = 1 \qquad \text{and} \qquad M_2 = 1$$

Now differentiate $(a)$:

$$3D^2 + 1 \equiv L_2 + (L_1D + M_1)(2D + 1) + (D^2 + D + 1)L_1$$

In this, set $D^2 = -D - 1$

$$-3D - 3 + 1 \equiv L_2 + 2L_1(-D - 1) + L_1D + 2M_1D + M_1$$

or

$$-3D - 2 \equiv (-L_1 + 2M_1)D + (-2L_1 + M_1 + 1)$$

from which

$$-L_1 + 2M_1 = -3$$
$$-2L_1 + M_1 = -3$$

and

$$L_1 = 1 \qquad \text{and} \qquad M_1 = -1$$

We then have

$$\frac{D^3 + D}{(D^2 + D + 1)^2} \equiv \frac{D + 1}{(D^2 + D + 1)^2} + \frac{D - 1}{D^2 + D + 1}$$

(30) The following are examples in partial fractions, to be worked out by the student:

I. (a) $\dfrac{D^2 + 11D + 14}{(D + 3)(D^2 - 4)}$. (b) $\dfrac{2D + 11}{(D - 2)(D + 3)}$.

(c) $\dfrac{6D - 1}{(2D + 1)(3D - 1)}$.

(d) $\dfrac{4D}{(D + 1)(D + 2)(D + 3)}$.

(e) $\dfrac{D^2 + 2D + 3}{(D - 1)(D - 2)(D - 3)(D - 4)}$. (f) $\dfrac{8D + 2}{D - D^3}$.

(g) $\dfrac{D^3 - D^2 - 5D + 4}{D^2 - 3D + 2}$.

II. (a) $\dfrac{D + 1}{D^2(D - 1)^2}$. (b) $\dfrac{D^2 - 1}{(D - 2)^3(D + 2)^2}$.

(c) $\dfrac{2D^2 - D + 1}{(D^2 - D)^2}$.

(d) $\dfrac{2D^3 - 3D^2 + 4D - 5}{(D + 3)^5}$. (e) $\dfrac{2D^3 - D^2 + 1}{(D - 2)^4}$.

III. (a) $\dfrac{1}{(D^2 + 1)(D^2 + 4)}$. (b) $\dfrac{D^2 + D + 1}{(D^2 + 1)(D^2 + 2)}$.

(c) $\dfrac{D^3 + D + 3}{D^4 + D^2 + 1}$.

IV. (a) $\dfrac{1}{(D^2 + 2)^3}$. (b) $\dfrac{2D^5 - D + 1}{(D^2 + D + 1)^3}$.

I, II. (a) $\dfrac{2D^2}{(D + 2)^2(D - 2)}$. (b) $\dfrac{3D - 4}{D(D - 4)^2}$.

(c) $\dfrac{D^2}{(D - 1)^3(D + 1)}$.

(d) $\dfrac{2D^2 - 3D - 2}{D(D - 1)^2(D + 3)^2}$. (e) $\dfrac{D^2 + 6D - 1}{(D - 3)^2(D - 1)}$.

I, III. (a) $\dfrac{3D - 1}{(D - 2)(D^2 + 1)}$. (b) $\dfrac{4D^2 - 3}{(D - 2)(D^2 + 2D + 5)}$.

(c) $\dfrac{1}{(D - 1)(D^2 + 1)}$.

(d) $\dfrac{D^2+2}{1+D^3}$. (e) $\dfrac{3D^2-D+2}{(D^2+2)(D^2-D-2)}$.

I, IV. (a) $\dfrac{D^2+2}{(1+D)^3}$. (b) $\dfrac{1}{(D-1)(D^2+1)^2}$.

II, III. (a) $\dfrac{D^3+1}{(D-1)^2(D^2+1)}$. (b) $\dfrac{D^3+2D}{(D-1)^5(D^2+1)}$.

## §5. Fundamental Theorems.

(1) In this section are given the fundamental working theorems of functions of the operator $D$. They are few in number but should be learned thoroughly by the student. They are merely formal but very powerful for the simplification of work. They are the following:

I. $F(D)\cdot e^{\phi(x)} \equiv e^{\phi(x)}\cdot F[D+\phi'(x)]$

II. $F[x+\phi'(D)]\cdot e^{\phi(D)} \equiv e^{\phi(D)}\cdot F(x)$

III. $F(D^2)\left.\begin{matrix}\sin\\ \cos\end{matrix}\right]ax = F(-a^2)\left.\begin{matrix}\sin\\ \cos\end{matrix}\right]ax$

IV. $F(D^2)\left.\begin{matrix}\sinh\\ \cosh\end{matrix}\right]ax = F(a^2)\left.\begin{matrix}\sinh\\ \cosh\end{matrix}\right]ax$

V. $$D^n\left(\prod_{i=1}^{m} u_i\right) \equiv \left[\sum_{i=1}^{m} D_i\right]^n\cdot\left(\prod_{k=1}^{m} u_k\right)$$
$$\equiv \sum_{i=0}^{n}\frac{n!}{\prod_{k=1}^{m} s_k!}\cdot\prod_{k=1}^{m} D_k^{s_i}u_k$$

VI. $$F(D) \equiv \sum_{k=0}^{\infty}\frac{1}{k!}D_1^k\cdot F^{(k)}(D_2) \equiv e^{D_1\cdot\frac{d}{dD_2}}\cdot F(D_2)$$

(2) $$F(D)\cdot e^{\phi(x)} \equiv e^{\phi(x)}F[D+\phi'(x)] \qquad \text{(I)}$$

The derivation of this theorem is given in Appendix III [§60 (11$a$) Eq. V] originally made by Robert Murphy in 1837. In 1853, it was again derived in an entirely different manner by Charles Graves [Appendix III, §62 (10) Eq. (15)]. A third derivation, by induction, is here given, as the simplest.

Given $S = S(x)$, then

$$\begin{aligned} D\cdot e^{\phi(x)}\cdot S &= e^{\phi(x)}\cdot DS + S\cdot De^{\phi(x)} \\ &= e^{\phi(x)}DS + e^{\phi(x)}\phi'(x)\cdot S \\ &= e^{\phi(x)}[D+\phi'(x)]\cdot S \end{aligned}$$

From this, by abstraction, we have

$$D \cdot e^{\phi(x)} \equiv e^{\phi(x)}[D + \phi'(x)] \tag{1}$$

Now using Eq. (1) and operating on the left by $e^{-\phi(x)}$, we obtain

$$e^{-\phi(x)} \cdot D \cdot e^{\phi(x)} \equiv D + \phi'(x) \tag{2}$$

With (2) and the axiom that equals multiplied by equals give equals, we have

$$e^{-\phi(x)} \cdot D \cdot e^{\phi(x)} \cdot e^{-\phi(x)} \cdot D \cdot e^{\phi(x)} \equiv [D + \phi'(x)] \cdot [D + \phi'(x)]$$

or

$$e^{-\phi(x)} \cdot D^2 \cdot e^{\phi(x)} \equiv [D + \phi'(x)]^2$$

Similarly,

$$e^{-\phi(x)} \cdot D^n \cdot e^{\phi(x)} \equiv [D + \phi'(x)]^n$$

By the use of constant multipliers with the foregoing, we can build up a rational integral function which will give

$$e^{-\phi(x)} \cdot F(D) \cdot e^{\phi(x)} \equiv F[D + \phi'(x)]$$

or

$$F(D) \cdot e^{\phi(x)} \equiv e^{\phi(x)}F[D + \phi'(x)] \tag{3}$$

the desired theorem.

(3) Or we can start with

$$D \cdot e^{\phi(x)} \cdot S = e^{\phi(x)}[D + \phi'(x)] \cdot S$$

and operate on the left by $D$, obtaining

$$D^2 e^{\phi(x)} S = D \cdot e^{\phi(x)}[D + \phi'(x)]S$$

The operation on the right is performed as follows:

$$\begin{aligned} D \cdot e^{\phi(x)}[D + \phi'(x)] \cdot S & \\ &= e^{\phi(x)} \cdot D[D + \phi'(x)]S + [D + \phi'(x)]S \cdot De^{\phi(x)} \\ &= e^{\phi(x)} \cdot D \cdot [D + \phi'(x)]S + e^{\phi(x)}\phi'(x)[D + \phi'(x)] \cdot S \\ &= e^{\phi(x)}[D + \phi'(x)][D + \phi'(x)] \cdot S \\ &= e^{\phi(x)}[D + \phi'(x)]^2 \cdot S \end{aligned}$$

so that

$$D^2 \cdot e^{\phi(x)} \cdot S = e^{\phi(x)}[D + \phi'(x)]^2 \cdot S$$

or

$$D^2 \cdot e^{\phi(x)} \equiv e^{\phi(x)}[D + \phi'(x)]^2$$

Similarly,

$$D^n \cdot e^{\phi(x)} \equiv e^{\phi(x)}[D + \phi'(x)]^n$$

and by multiplying by constants and adding for various values of $n$ we obtain Eq. (3).

(4) For the inverse, use Eq. (3):

$$F(D) \cdot e^{\phi(x)} \equiv e^{\phi(x)} \cdot F[D + \phi'(x)]$$

then operate on the left by $F^{-1}(D)$ and on the right by

$$F^{-1}[D + \phi'(x)];$$

thus:

$$\begin{aligned} F^{-1}(D) \cdot F(D)e^{\phi(x)} \cdot F^{-1}[D + \phi'(x)] \\ \equiv F^{-1}(D) \cdot e^{\phi(x)}F[D + \phi'(x)] \cdot F^{-1}[D + \phi'(x)] \end{aligned}$$

Since $F^{-1} \cdot F \equiv 1$, we then have

$$F^{-1}(D)e^{\phi(x)} \equiv e^{\phi(x)}F^{-1}[D + \phi'(x)] \tag{Ia}$$

There is no question as to infinities here involved, as since $F(D)$ is not zero, $F[D + \phi'(x)]$ is not.

(5) If $\phi(x) \equiv ax$, we have $\phi'(x) \equiv a$, and Theorem I becomes

$$F(D)e^{ax} \equiv e^{ax}F(D + a) \tag{Ib}$$

and if a subject $S \equiv 1$ is used with

$$F(D + a) \equiv F(a) + F'(a)D + \tfrac{1}{2}F''(a)D^2 + \cdots,$$

it is seen that

$$\begin{aligned} F(D)e^{ax} \cdot 1 &\equiv e^{ax}F(D + a)1 \\ &\equiv e^{ax}[F(a) + F'(a)D + \cdots]1 \\ &= e^{ax}F(a) \end{aligned}$$

*i.e.*,

$$F(D)e^{ax} = e^{ax}F(a) \tag{Ic}$$

(6) If $S \equiv 1$ and $\phi(x) \equiv ax$, and $a$ is a root of multiplicity $r$ of $F(D) = 0$, Theorem I$a$ gives us an interesting form of practical value in applications; thus:

$$F^{-1}(D)e^{ax} \cdot 1 = e^{ax}F^{-1}(D + a)1$$

in which $F(D + a)$ is expanded by Taylor's theorem to obtain

$$F(D + a) \equiv \sum_{k=r}^{\infty} \frac{1}{k!} F^{(k)}(a) \cdot D^k$$

Then

$$F^{-1}(D + a) \cdot 1 = \frac{1}{\sum_{k=r}^{\infty} F^{(k)}(a) \cdot \frac{D^k}{k!}} 1$$

We can neglect terms $k > r$, obtaining

$$F^{-1}(D + a) \cdot 1 = \frac{r!}{F^{(r)}(a)} \cdot \frac{1}{D^r} 1 = \frac{x^r}{F^{(r)}(a)}$$

or

$$F^{-1}(D) \cdot e^{ax} = \frac{x^r}{F^{(r)}(a)} e^{ax} \qquad \text{(I}d\text{)}$$

The justification for this neglect of the terms of power higher than $r$ is shown by the following. If $a$ is a root of $F(D) = 0$ of multiplicity $r$, then

$$F(D) \equiv (D - a)^r \phi(D)$$

and

$$F^{(r)}(D) \mid_{D=a} = r! \cdot \phi(D) \mid_{D=a}$$

We then have

$$\begin{aligned} F^{-1}(D) \cdot e^{ax} \cdot 1 &= \frac{1}{\phi(D)(D - a)^r} e^{ax} \cdot 1 && \text{by (5)} \\ &= e^{ax} \cdot \frac{1}{D^r \cdot \phi(D + a)} \cdot 1 \\ &= e^{ax} \frac{1}{D^r} \cdot \frac{1}{\phi(a)} 1 && \text{by (6)} \\ &= e^{ax} \frac{1}{\phi(a)} \cdot \frac{1}{D^r} 1 \\ &= e^{ax} \frac{1}{\phi(a)} \frac{x^r}{r!} \\ &= e^{ax} \frac{x^r}{F^{(r)}(a)} \end{aligned}$$

(7) $$e^{\phi(D)} \cdot F(x) \equiv F[x + \phi'(D)] \cdot e^{\phi(D)} \qquad \text{(II)}$$

This correlative theorem to Theorem I we shall derive by induction, as follows: Using $F(D) \equiv D$, in I,

$$e^{-\phi(x)} \cdot D \cdot e^{\phi(x)} \equiv D + \phi'(x)$$

and in it replace $D$ by $-x$ and $x$ by $D$ [according to Graves's theorem, Appendix III, §62 (2) Eq. (3)].

$$e^{-\phi(D)}(-x)e^{\phi(D)} \equiv -x + \phi'(D)$$

or

$$e^{-\phi(D)} \cdot x \cdot e^{\phi(D)} \equiv x - \phi'(D)$$

Now, by the axiom of multiplying equals,

$$e^{-\phi(D)} \cdot x \cdot e^{\phi(D)} \cdot e^{-\phi(D)} \cdot x \cdot e^{\phi(D)} \equiv [x - \phi'(D)] \cdot [x - \phi'(D)]$$

or

$$e^{-\phi(D)} \cdot x^2 \cdot e^{\phi(D)} \equiv [x - \phi'(D)]^2$$

Similarly,

$$e^{-\phi(D)} \cdot x^n \cdot e^{\phi(D)} \equiv [x - \phi'(D)]^n$$

and by multiplication by constants and adding for various values of $n$ we obtain

$$e^{-\phi(D)} \cdot F(x) \cdot e^{\phi(D)} \equiv F[x - \phi'(D)].$$

Now multiply on the right by $e^{-\phi(D)}$, obtaining

$$e^{-\phi(D)} \cdot F(x) \equiv F[x - \phi'(D)]e^{-\phi(D)}$$

Then insert $\phi(D)$ for $-\phi(D)$ and get the theorem desired. Notice that when the $e^{\phi(D)}$ is taken across the $F(x)$ to the right, it adds $\phi'(D)$ to $x$, whereas in Theorem I when the $e^{\phi(x)}$ is taken across the $F(D)$ to the right it subtracts $\phi'(x)$. This Theorem II was first exhibited in 1853 by Charles Graves [Appendix III, §62 (11) Eq. (14)]. Another proof of II can be obtained by the use of the extension of Leibnitz's theorem below.

(8) For the inverse, operate on II on the left by $F^{-1}[x + \phi'(D)]$ and on the right by $F^{-1}(x)$; thus:

$$F[x + \phi'(D)]e^{\phi(D)} \equiv e^{\phi(D)} \cdot F(x)$$

$$F^{-1}[x + \phi'(D)]F[x + \phi'(D)]e^{\phi(D)}F^{-1}(x)$$
$$\equiv F^{-1}[x + \phi'(D)]e^{\phi(D)}F(x)F^{-1}(x)$$

Then with $FF^{-1} \equiv F^{-1}F \equiv 1$, we shall have

$$e^{\phi(D)}F^{-1}(x) \equiv F^{-1}[x + \phi'(D)]e^{\phi(D)} \qquad \text{(II}a\text{)}$$

(9) Some special cases of II are interesting and useful. Set $\phi(D) \equiv kD$; then $\phi'(D) \equiv k$, and II becomes

$$e^{kD}F(x) \equiv F(x + k)e^{kD} \qquad \text{(II}b\text{)}$$

Now, if the subject $S \equiv 1$, we have

$$e^{kD}F(x) \cdot 1 = F(x + k) \cdot e^{kD} \cdot 1$$

Perform the operation on the right after expanding

$$e^{kD} \equiv 1 + kD + \frac{k^2}{2}D^2 + \cdots$$

so that

$$e^{kD}1 = \left(1 + kD + \frac{k^2}{2}D^2 + \cdots\right)1 = 1$$

which gives us

$$e^{kD}F(x) = F(x + k) \qquad \text{(II}c\text{)}$$

which is Taylor's theorem in one variable in symbolical form.

(10) If $e^k = a$, so that $k = \log_e a$, then II$b$ becomes

$$a^DF(x) \equiv F(x + \log_e a)a^D \qquad \text{(II}d\text{)}$$

from which, with $S \equiv 1$, we obtain

$$a^DF(x) = F(x + \log_e a) \qquad \text{(II}e\text{)}$$

(11) Owing to the fact that an even number of differentiations of sin $ax$, cos $ax$, sinh $ax$, and cosh $ax$ bring in the same functions again with a multiplicative constant factor of a given form, we have a simple substitution theorem for operators on them, as follows:

$$F(D^2)\left.\begin{matrix}\sin\\ \cos\end{matrix}\right]ax = F(-a^2)\left.\begin{matrix}\sin\\ \cos\end{matrix}\right]ax \qquad \text{(III)}$$

$$F(D^2)\left.\begin{matrix}\sinh\\ \cosh\end{matrix}\right]ax = F(a^2)\left.\begin{matrix}\sinh\\ \cosh\end{matrix}\right]ax \qquad \text{(IV)}$$

These are inductively proved without any difficulty. The proofs should be worked by the student.

(12) A question involved here is that of so changing the form of a function that $D^2$ appears in it for substitution. Given a function $f(D)$, can we form the function $\phi(D^2) \equiv f(D)$? An example or two will show that any function can be put into the following form:

$$f(D) = \phi(D^2) + D \cdot \psi(D^2)$$

*i.e.*,

(*a*) $D^2 + 2D - 1 \equiv (D^2 - 1) + D \cdot 2$

(*b*) $D^5 + 3D^4 - 2D^3 + D^2 - D + 1 \equiv (3D^4 + D^2 + 1) + D(D^4 - 2D^2 - 1)$

It is also easy to see that any form can be factored or transformed in such a way that one or more factors may be forms in $D^2$, as

$$f(D) \equiv \phi(D^2) \cdot \psi(D)$$

*i.e.*,

(*a*) $\dfrac{1}{D-1} \equiv \dfrac{1}{D-1} \cdot \dfrac{D+1}{D+1} = \dfrac{1}{D^2-1}(D+1)$

(*b*) $$\frac{1}{D^2 - D + 1} \equiv \frac{1}{(D^2+1) - D} \equiv \frac{1}{(D^2+1) - D} \cdot \frac{(D^2+1) + D}{(D^2+1) + D} \equiv \frac{(D^2+1) + D}{(D^2+1)^2 - D^2} \equiv \frac{D^2+1}{(D^2+1)^2 - D^2} + \frac{1}{(D^2+1)^2 - D^2} D$$

This latter type of transformation of operators is especially useful in the solution of differential equations.

(13) For the inverses of III and IV it must be noted that there are exceptions to the validity of the results whenever $F(\pm a^2) \equiv 0$. In such cases, the operations upon the trigonometric functions cannot be directly performed unless substitutions of the exponential functions are made before operating. Then Theorem I will control and give valid results.

(14) Examples of the direct and the inverse operations of the foregoing type, and the exception, are here given.

*a*. Direct:

$$D^4 \cdot \sin 3x = (D^2)^2 \cdot \sin 3x = (-3^2)^2 \sin 3x = 81 \sin 3x$$

*b.* Direct:

$$(D^4 - 2D^2 + 3)\cdot \sin x = [(-1)^2 - 2(-1) + 3] \sin x$$
$$= 6 \sin x$$

*c.* Inverse:

$$\frac{1}{D^2+1}\cos 2x = \frac{1}{-2^2+1}\cos 2x = -\frac{1}{3}\cos 2x$$

*d.* Inverse:

$$\begin{aligned}
\frac{1}{D+1}\sin 3x &= \frac{1}{D+1}\,\frac{D-1}{D-1}\sin 3x \\
&= (D-1)\frac{1}{D^2-1}\sin 3x \\
&= (D-1)\frac{1}{-3^2-1}\sin 3x \\
&= -\tfrac{1}{10}(D-1)\sin 3x \\
&= -\tfrac{1}{10}[3\cos 3x - \sin 3x]
\end{aligned}$$

*e.* Inverse; exception:

$$\frac{1}{D^2+1}\sin x = \frac{1}{-1+1}\sin x = \infty \cdot \sin x$$

To work this we must perform the operation

$$\begin{aligned}
\frac{1}{D^2+1}e^{ix} &= e^{ix}\frac{1}{(D+i)^2+1}1 && \text{[Th. I}b\text{]} \\
&= e^{ix}\frac{1}{D^2+2iD}1 && \text{[Expanding]} \\
&= e^{ix}\frac{1}{D}\,\frac{1}{D+2i}1 && \text{[Factoring]} \\
&= e^{ix}\frac{1}{D}\cdot\frac{1}{2i} && \text{[Th. I}c\text{]} \\
&= e^{ix}\frac{1}{2i}x. && \text{[Def. of } D^{-1}\text{]}
\end{aligned}$$

Now substitute

$$e^{ix} \equiv \cos x + i \sin x$$

and obtain

$$\begin{aligned}
\frac{1}{D^2+1}[\cos x + i\sin x] &= \frac{x}{2i}[\cos x + i \sin x] \\
&= \frac{x\sin x}{2} + i\frac{x}{-2}\cos x
\end{aligned}$$

then equate real and imaginary terms;

$$\frac{\phantom{1}}{D^2+1}\cos x = x \sin x$$

$$\frac{1}{D^2+1}\sin x = -\frac{x}{2}\cos x$$

The latter is the desired result in this exceptional case. It is thus evident that the exceptional case brings an integration into the result.

(15) In general, when $F(-a^2) = 0$, we may note the following:

$$\frac{1}{F(D^2)}\left.\begin{matrix}\sin\\ \cos\end{matrix}\right]ax \equiv \frac{1}{F(D^2)}\cdot\frac{e^{iax} \mp e^{-iax}}{2}$$

$$= \frac{1}{2}\left[\frac{1}{F(D^2)}\cdot e^{iax} \mp \frac{1}{F(D^2)}\cdot e^{-iax}\right]$$

$$= \frac{1}{2}\left[e^{iax}\frac{1}{F[(D+ia)^2]}1 \mp e^{-iax}\frac{1}{F[(D-ia)^2]}1\right]$$

$F[(D \pm ia)^2]$ will be factorable and of form $D^n \cdot \phi(D \pm ia)$, so that the operations on 1 will bring results as in

$$\frac{1}{F[(D \pm ia)^2]}1 = \frac{1}{D^n}\frac{1}{\phi(D \pm ia)}1$$

$$= \frac{1}{D^n}\frac{1}{\phi(\pm ia)}1$$

$$= \frac{1}{\phi(\pm ia)}\cdot\frac{1}{D^n}1 = \frac{1}{\phi(\pm ia)}\cdot\frac{x^n}{n!}$$

The final result will then be

$$\frac{1}{F(D^2)}\left.\begin{matrix}\sin\\ \cos\end{matrix}\right]ax = \frac{1}{2}\frac{x^n}{n!}\left[\frac{1}{\phi(ia)}e^{iax} \mp \frac{1}{\phi(-ia)}e^{-iax}\right]$$

$$= \frac{x^n}{2\cdot n!}\left[\frac{1}{\phi(ia)}(\cos ax + i\sin ax) \mp \right.$$

$$\left.\frac{1}{\phi(-ia)}(\cos ax - i \sin ax)\right]$$

$$= \frac{x^n}{2\cdot n!}\left[\left(\frac{1}{\phi(ia)} \mp \frac{1}{\phi(-ia)}\right)\cos ax\right.$$

$$\left. + i\left(\frac{1}{\phi(ia)} \mp \frac{1}{\phi(-ia)}\right)\sin ax\right]$$

The real term of this result only is to be taken. There will always be a real term.

(16) Examples to be worked by the student:

(a) $\dfrac{1}{D^2 + \omega^2} k \cos \omega x$

(b) $\dfrac{1}{D^2 + p^2}(-g \sin px)$

(c) $\dfrac{1}{D^2 + n^2} \sin nx$

(d) $\dfrac{1}{D^4 - 2D^2 + 1} \cos x$

(e) $\dfrac{1}{D^2 - 1} \cosh x$

(f) $\dfrac{1}{D^2 + 4} \cos 2x$

(17) *The Leibnitz Theorem.* Stated in Appendix I, §57 (33), this theorem will be used here as the foundation for one that is the culmination of fundamental theorems in this subject and has a remarkably widespread application. First let us develop it as Leibnitz did with $w = u \cdot v$.

$$\begin{aligned}
w &= u \cdot v \\
dw &= u \cdot dv + v \cdot du \\
d^2w &= u \cdot d^2v + dv \cdot du + v \cdot d^2u + du \cdot dv \\
&= u \cdot d^2v + 2du \cdot dv + v \cdot d^2u \\
d^3w &= u \cdot d^3v + d^2v \cdot du + 2du \cdot d^2v + 2 \cdot dv \cdot d^2u \\
&\qquad + v \cdot d^3u + d^2u \cdot dv \\
&= u \cdot d^3v + 3du \cdot d^2v + 3d^2u \cdot dv + v \cdot d^3v
\end{aligned}$$

$$\begin{aligned}
d^nw &= u \cdot d^nv + n \cdot du \cdot d^{n-1}v + \frac{n(n-1)}{2} d^2u \cdot d^{n-2}v + \cdots \\
&\qquad + \frac{n(n-1)}{2} d^{n-2}u \cdot d^2v + n \cdot d^{n-1}u \cdot dv + v \cdot d^nu
\end{aligned}$$

or

$$\begin{aligned}
d^n(u \cdot v) &= u \cdot d^nv + C_1du \cdot d^{n-1}v + C_2d^2u \cdot d^{n-2}v + \cdots \\
&\qquad + C_2d^{n-2}u \cdot d^2v + C_1d^{n-1}u \cdot dv + vd^nu
\end{aligned}$$

where $C_1$, $C_2$, . . . etc., are the binomial coefficients.

(18) If, now, $u$ and $v$ are functions of $x$, we can write

$$\frac{d^n(u \cdot v)}{dx^n} = u\frac{d^n}{dx^n}v + C_1\frac{du}{dx}\frac{d^{n-1}}{dx^{n-1}}v + \cdots$$

or, symbolically,

$$D^n \cdot (u \cdot v) = uD^nv + C_1Du \cdot D^{n-1}v + \cdots$$

*i.e.*,

$$D^n \cdot (u \cdot v) = \sum_{k=0}^{n} {}_nC_k D^k u \cdot D^{n-k} v, \qquad {}_nC_k = \frac{n!}{(n-k)!k!} \qquad \text{(V}a\text{)}$$

(19) Theorem V*a* can be derived symbolically by a device that is used considerably in operational work with success and can be stated as follows:

**A linear operator can be separated into a sum of linear operators each one of which is of the same type as the whole operator but is limited in its capacity to a particular subject or factor of a subject.**

That is to say, if $D \equiv \frac{d}{dx}$, we can write

$$D \equiv D_1 + D_2 + \cdots$$

where $D_1 \equiv \frac{d}{dx}$ and $D_2 \equiv \frac{d}{dx}$, but $D_1$ will differentiate with respect to $x$ one of the factors of the subject and be null as respects all others, $D_2$ differentiates another factor, etc. Thus, with

$$\begin{aligned} S &\equiv S_1(x) \cdot S_2(x) \\ D &\equiv D_1 + D_2 \\ D \cdot S &\equiv (D_1 + D_2) \cdot S_1(x) \cdot S_2(x) \\ &\equiv D_1 \cdot S_1(x) \cdot S_2(x) + D_2 \cdot S_1(x) \cdot S_2(x) \\ &\equiv S_2(x) \cdot D \cdot S_1(x) + S_1(x) \cdot D \cdot S_2(x) \end{aligned}$$

now,

$$D^n \equiv (D_1 + D_2)^n \equiv \sum_{k=0}^{n} {}_nC_k D_1{}^k \cdot D_2{}^{n-k}$$

and if this is applied to $S \equiv u \cdot v$, we may have $D_1$ operating only on $u$ and null as regards $v$ and $D_2$ operating only on $v$ and null as regards $u$. Then

$$\begin{aligned} D^n \cdot (u \cdot v) &= \sum_{k=0}^{n} {}_nC_k D_1{}^k \cdot D_2{}^{n-k} \cdot u \cdot v \\ &= \sum_{k=0}^{n} {}_nC_k D_1{}^k u \cdot D_2{}^{n-k} v \\ &= \sum_{k=0}^{n} {}_nC_k D^k u \cdot D^{n-k} v, \text{ the former theorem} \end{aligned}$$

(20) In the literature this has been extended to a product of three factors:

$$D^n \cdot (u \cdot v \cdot w) \equiv (D_1 + D_2 + D_3)^n \cdot u \cdot v \cdot w \qquad \text{(V}b\text{)}$$

Expand $(D_1 + D_2 + D_3)^n$ by the binomial theorem and operate $D_1$ on $u$, $D_2$ on $v$, and $D_3$ on $w$ only.

(21) It is obvious that this can be extended to any number of factors; thus:

$$D^n\left(\prod_{k=1}^{m} u_k\right) \equiv \left[\sum_{k=1}^{m} D_k\right]^n \cdot \left(\prod_{k=1}^{m} u_k\right) \text{ symbolically}$$

$$\equiv \sum \frac{n!}{\prod_{i=1}^{m} s_i!} \prod_{k=1}^{m} D_k^{s_i} u_k \qquad \text{(V)}$$

where $\Sigma s_i = m$, $D_k \sim u_k$, by the multinomial theorem.

(22) *Extension of the Leibnitz Theorem.* The theorem developed in this paragraph was first stated as an easily developed one by Hargreave in 1847 (*Phil. Trans., London,* 1848) without proof. The Rev. Robert Carmichael, who wrote "A Treatise on the Calculus of Operations" in 1855, developed it by induction, as follows:

Use, as above, the identical operators

$$D^n \equiv \sum_{k=0}^{n} {}_nC_k D_1^k D_2^{n-k}$$

Multiply by the constant $a_n$, take the equation for all integral values of $n$, and add to obtain on the left an integral rational function of $D$. We then have

$$F(D) \equiv \sum_{k=0}^{\infty} \frac{1}{k!} D_1^k F^{(k)}(D_2) \qquad \text{(VI)}$$

(23) A simpler way to derive this is by means of Taylor's theorem for one variable. With

$$F(x + h) \equiv \sum_{k=0}^{\infty} \frac{h^k}{k!} F^{(k)}(x)$$

set

$$x + h \equiv D \equiv D_1 + D_2, \qquad x \equiv D_2, \qquad h \equiv D_1$$

then, at once

$$F(D) \equiv \sum_{k=0}^{\infty} \frac{D_1{}^k}{k!} F^{(k)}(D_2)$$

(24) If we use the symbolical form of Taylor's theorem, *viz.*,

$$F(x + h) \equiv e^{hD} \cdot F(x)$$

and set

$$D \equiv \frac{u}{dx} \equiv x + h \equiv p + q \equiv \frac{u}{dp}$$
$$x \equiv p, \qquad h \equiv q$$

then

$$F(D) \equiv e^{q\frac{d}{dp}} F(p)$$

Expand the right-hand side, which then becomes identical with the right-hand side of (VI); thus:

$$\begin{aligned} F(D) &\equiv e^{q\frac{d}{dp}} F(p) \\ &\equiv \left[1 + q\frac{d}{dp} + \frac{q^2}{2!}\frac{d^2}{dp^2} + \qquad \right. F(p) \\ &\equiv \sum_{k=0}^{\infty} \frac{1}{k!}\, q^k \left[\frac{d^k}{dp^k} F(p)\right. \\ &\equiv \sum_{k=0}^{\infty} \frac{1}{k!}\, q^k F^{(k)}(p) \qquad \text{of form} \qquad \sum_{k=0}^{\infty} \frac{1}{k!} D_1{}^k F^{(k)}(D_2) \end{aligned}$$

In this, $q$ and $p$ are differentiators, and each can operate independently in similar manner as $D_1$ and $D_2$. This equation would be used as follows:

$$F(D) \cdot P(x) \cdot Q(x) = \sum_{k=0}^{\infty} \frac{1}{k!} q^k \cdot F^{(k)}(p) \cdot P \cdot Q$$

where $q$ operates only on $Q$, and $p$ only on $P$, and $F^{(k)}(p)$ means the form in $p$ obtained by $k$ successive differentiations of $F(p)$ with respect to $p$.

(25) An example of the use of (VI) is

$$\frac{1}{D+1}x^3 \sin 2x$$

Here

$$F(D) \equiv \frac{1}{D+1} \qquad Dx^3 = 3x^2$$
$$F'(D) \equiv \frac{-1}{(D+1)^2} \qquad D^2x^3 = 6x$$
$$F''(D) \equiv \frac{2}{(D+1)^3} \qquad D^3x^3 = 6$$
$$F'''(D) = \frac{-6}{(D+1)^4} \qquad D^4x^3 = 0$$

There will thus be only four terms in the expansion.

$$\frac{1}{D+1}x^3 \sin 2x = x^3\frac{1}{D+1}\sin 2x - 3x^2\frac{1}{(D+1)^2}\sin 2x + \frac{6x}{2}\frac{1}{(D+1)^3}\sin 2x - \frac{6}{6}\frac{1}{(D+1)^4}\sin 2x$$

In this, the integrations are only of $\sin 2x$, as follows:

$$\frac{1}{D+1}\sin 2x = \frac{D-1}{D^2-1}\sin 2x = \frac{1}{-5}(D-1)\sin 2x$$
$$= -\tfrac{1}{5}(2\cos 2x - \sin 2x)$$

$$\frac{1}{(D+1)^2}\sin 2x = \frac{1}{D^2+2D+1}\sin 2x = \frac{1}{-3+2D}\sin 2x$$
$$= \frac{2D+3}{4D^2-9}\sin 2x = \frac{1}{-25}(2D+3)\sin 2x$$
$$-\tfrac{1}{25}(3\sin 2x + 4\cos 2x)$$

$$\frac{1}{(D+1)^3}\sin 2x = \frac{1}{D^3+3D^2+3D+1}\sin 2x = \frac{1}{-D-11}\sin 2x$$
$$= \frac{11-D}{D^2-121}\sin 2x = \frac{1}{-125}(11\sin 2x - 2\cos 2x)$$

$$\frac{1}{(D+1)^4}\sin 2x = \frac{1}{D^4+4D^3+6D^2+4D+1}\sin 2x$$
$$\frac{-1}{12D+7}\sin 2x = \frac{-(12D-7)}{144D^2-49}\sin 2x$$
$$= \tfrac{1}{625}(24\cos 2x - 7\sin 2x)$$

Collecting terms, we have the result

$$\frac{1}{D+1}x^3 \sin 2x = \left(-\frac{2}{5}x^3 + \frac{12}{25}x^2 + \frac{6}{125}x - \frac{24}{625}\right)\cos 2x$$
$$+ \left(+\frac{1}{5}x^3 + \frac{9}{25}x^2 - \frac{33}{125}x + \frac{7}{625}\right)\sin 2x$$

To perform the same operation analytically by the classical method would mean the use of integration by parts. Theorem VI really is the symbolical form of the extended theorem of integration by parts. The integral can readily be set up if we use Theorem I; thus:

$$\frac{1}{D+1} \equiv \frac{1}{D+1}e^{-x} \cdot e^x \equiv e^{-x}\frac{1}{D}e^x$$

and

$$\frac{1}{D+1}x^3 \sin 2x = e^{-x}\frac{1}{D}e^x x^3 \sin 2x$$
$$= e^{-x}\int e^x x^3 \sin 2x dx$$

Using successive integration by parts upon this gives the former result.

(26) Other examples to be solved by the student are

| | | |
|---|---|---|
| (a) $\frac{1}{D+1} x^2 \cos x$ | (f) $\frac{}{D^2+m^2} x \cos ax$ | (k) $\frac{1}{(D-2)^3} x^2e^2$ |
| (b) $\frac{1}{D-1} x \sin x$ | (g) $\frac{1}{D^3-1} x \sin x$ | (l) $\frac{1}{D} x^4e^x$ |
| (c) $\frac{1}{D^2} x^2 \sin x$ | (h) $\frac{1}{(D-1)^2} x^2e^3$ | (m) $D^5x^6e^{3x}$ |
| (d) $\frac{1}{D^2-1} x^2 \cos x$ | (i) $\frac{1}{D^2+1} xe^2$ | (n) $D^5x^3 \cos \frac{x}{2}$ |
| (e) $\frac{1}{D^2+4} x \sin x$ | (j) $\frac{1}{D^3} xe^x$ | (o) $D^3x^3 \log 2x$ |

(27) When $F(D)$ is an inverse, such as $\frac{1}{f(D)}$, Theorem VI sometimes needs to be used with great care. The care consists in the proper ordering of the factors in the derivatives of $F(D)$ which appear in the successive terms. This is not necessary in any of the problems of paragraph (26). But in all cases the results

should be carefully checked for validity by the proper inverse operations. If under such check the proper original function does not appear, then the result is incorrect and the operations should be attacked as follows.

(28) In the literature appears the following proved theorem:

$$\frac{1}{F(D)} \cdot x \cdot V = x \cdot \frac{1}{F(D)} \cdot V - \frac{1}{F(D)} \cdot F'(D) \cdot \frac{1}{F(D)} \cdot V. \quad \text{[Murray]}$$

Let us operationally build upon it. Abstract the subject and obtain the following operational identity

$$\frac{1}{F} \cdot x \equiv \left[x - \frac{1}{F} \cdot F'\right]\frac{1}{F}. \tag{$a$}$$

Now

$$\begin{aligned} \cdot x^2 &\equiv \left(\frac{1}{F} \cdot x\right) \cdot x \\ &\equiv \left[x - \frac{1}{F} \cdot F'\right]\frac{1}{F} \cdot x, \qquad \text{[by } (a)\text{]} \end{aligned}$$

and then by a second use of $(a)$ we have

$$\begin{aligned} \frac{1}{F} \cdot x^2 \equiv & \left[x - \frac{1}{F} \cdot F'\right]\left[x - \frac{1}{F} \cdot F'\right]\frac{1}{F} \\ & \left[x - \frac{1}{F} \cdot F'\right]^2 \cdot \frac{1}{F} \end{aligned}$$

Similarly,

$$\frac{1}{F} \cdot x^3 \equiv \left[x - \frac{1}{F} \cdot F'\right]^3 \cdot \frac{1}{F},$$

*i.e.*, generally

$$\frac{1}{F} \cdot x^n \equiv \left[x - \frac{1}{F} \cdot F'\right]^n \cdot \frac{1}{F}$$

Then inserting a subject $V$, which can be any function of $x$, we have a theorem which if properly used will produce correct results.

$$\frac{1}{F(D)}(x^n \cdot V) = \left[x - \frac{1}{F} \cdot F'\right]^n \cdot \frac{1}{F(D)} \cdot V \tag{VII}$$

(29) Examples in which this theorem is necessary:

(a) $\dfrac{1}{D^2 + 1} \cdot x \cdot \sin x$

(b) $\dfrac{1}{D^2 + 1} \cdot r^2 \cdot \sin x$

(c) $\dfrac{1}{D^2 + m^2} \; x \cdot \cos mx.$

**§6. Elementary Interpretation.**

(1) The student should refer to the definition of "interpretation" in this text [I §3 (7)]. The operator $D$ or any integral rational function of $D$ of course is readily interpreted when necessary for use in an operational form. It is simply the sign for differentiation. The inverses, however, from their very form must either be transformed by algebraic or calculus methods into direct operators or be known forms readily transformable directly into classic forms.

(2) All integral rational functions of $D$ can be separated into either linear real or linear complex factors, single or multiple. All inverses, therefore, of such functions can be separated into partial fractions [II §4 (17)]. It is well to know the classic parallel for each of the elementary forms that come under this head. We shall take each one in order and show by operational methods what those parallels are.

(3) $(D - \alpha)^{-1}$. Since $e^{\alpha x} \cdot e^{-\alpha x} \equiv 1$, we can operate on the right of this form by $e^{\alpha x} \cdot e^{-\alpha x}$ and with Theorem (I*b*) obtain a form that illuminates it strikingly.

$$(D - \alpha)^{-1} \equiv (D - \alpha)^{-1} e^{\alpha x} \cdot e^{-\alpha x} \equiv e^{\alpha x}(D + \alpha - \alpha)^{-1} e^{-\alpha x}$$
$$\equiv e^{\alpha x} D^{-1} e^{-\alpha x}$$

Now, since $D^{-1} \equiv \int ()dx$, we shall have

$$(D - \alpha)^{-1} \equiv e^{\alpha x} \int e^{-\alpha x}()dx, \text{ the classic form}$$

(4) Another approach to the interpretation of $(D - \alpha)^{-1}$ would be the following:

$$\frac{dy}{dx} - \alpha y \equiv (D - \alpha)y$$

Now, $\dfrac{dy}{dx} - \alpha y = f(x)$, we know, has the solution

$$= e^{\alpha x}\int e^{-\alpha x}f(x)dx + Ce^{\alpha x}$$

so we can assume that $(D - \alpha)y = f(x)$ also has that solution. The first part of this solution is exactly that obtained by the operational method:

$$\begin{aligned}(D - \alpha)y &= f(x)\\ y &= \frac{1}{D - \alpha}f(x) = (D - \alpha)^{-1}f(x)\\ &= e^{\alpha x}\int e^{-\alpha x}f(x)dx\end{aligned}$$

Notice that this latter does not include the $Ce^{\alpha x}$. In other words, only the particular integral is given by the operational form. The $Ce^{\alpha x}$ and its parallels for other operators will be discussed later.

(5) The form $(D - \alpha)^{-1} \equiv e^{\alpha x}\int e^{-\alpha x}()dx$ can be further transformed by changing the letter under the integral and carrying $e^{\alpha x}$ into it.

$$\begin{aligned}(D - \alpha)^{-1} &\equiv e^{\alpha x}\int^x e^{-\alpha u}()du\\ &\equiv \int^x e^{\alpha(x-u)}()du\end{aligned}$$

Operationally, this would be indicated as follows:

$$(D - \alpha)^{-1} \equiv e^{\alpha x}D^{-1}e^{-\alpha x}$$

where $\mu^{-1}$ is an operation with respect to $u$ in the result of which $x$ is to be put for $u$. The subject must be one in $u$.

(6) $(D - \alpha)^{-n}$. By the same processes as in (5) we can transform $(D - \alpha)^{-n}$:

$$\begin{aligned}(D - \alpha)^{-n} &\equiv (D - \alpha)^{-n}e^{\alpha x}\cdot e^{-\alpha x} \equiv e^{\alpha x}D^{-n}e^{-\alpha x}\\ &\equiv e^{\alpha x}\mu^{-n}e^{-\alpha u} \equiv \mu^{-n}e^{\alpha(x-u)}\\ &\equiv \int^x\int^u \cdot\cdot\cdot \int^u e^{\alpha(x-u)}()du^n\end{aligned}$$

(7) By means of a well-known theorem we can transform this into a single integral; thus: since

$$\int_{x_0}^{x} \cdots \int_{x_0}^{x} F(x)dx^n \equiv \int \frac{1}{(n-1)!}(x-u)^{n-1}F(u)du^*$$

we shall have

$$\int_{x_0}^{x} \cdots \int_{x_0}^{x} e^{-\alpha x}f(x)dx^n \equiv \frac{1}{(n-1)!}\int_{x_0}^{x}(x-u)^{n-1}e^{-\alpha u}f(u)du$$

and then

$$(D-\alpha)^{-n} \equiv \frac{1}{(n-1)!}\mu^{-1}(x-u)^{n-1}e^{\alpha(x-u)}$$
$$\equiv \frac{1}{(n-1)!}\int^{x}(x-u)^{n-1}e^{\alpha(x-u)}(\,)du$$

(8) $\prod_{i=1}^{n}(D-\alpha_i)^{-1}$. Separate this form into partial fractions (Type I) and obtain a sum of forms of type $(D-\alpha)^{-1}$ as covered in (3); *i.e.*:

$$\prod_{i=1}^{n}(D-\alpha_i)^{-1} \equiv \sum_{i=1}^{n} A_i(D-\alpha_i)^{-1} \qquad \text{[II §4 (20)]}$$
$$\equiv \sum_{i=1}^{n} A_i e^{\alpha_i x}D^{-1}e^{-\alpha_i x} \qquad \text{[Th. I]}$$
$$\equiv \sum_{i=1}^{n} A_i \mu^{-1}e^{\alpha_i(x-u)}$$
$$\equiv \sum_{i=1}^{n} A_i \int^{x} e^{\alpha_i(x-u)}(\,)du$$

Or, if we prefer to interpret it as a multiple integral, proceed as follows:

* Goursat-Hedrick-Dunkel, "Differential Equations," §18, p. 36, Boston, 1917.

$$\prod_{i=1}^{n}(D-\alpha_i)^{-1} \equiv \prod_{i=1}^{n}[(D-\alpha_i)^{-1}e^{\alpha_i x}e^{-\alpha_i x}]$$

$$\equiv \prod_{i=1}^{\cdot\cdot} e^{\alpha_i x}D^{-1}e^{-\alpha_i x}$$

$$\equiv (e^{\alpha_1 x}D^{-1}e^{-\alpha_1 x})(e^{\alpha_2 x}D^{-1}e^{-\alpha_2 x}) \cdots (e^{\alpha_{n-1} x}D^{-1}e^{-\alpha_{n-1} x})(e^{\alpha_n x}D^{-1}e^{-\alpha_n x})$$

$$\equiv \mu_0{}^{-1}e^{\alpha_1(x-u_1)}\mu_1{}^{-1}e^{\alpha_2(u_1-u_2)} \cdots \mu_{n-2}{}^{-1}e^{\alpha_{n-1}(u_{n-2}-u_{n-1})}\mu_{n-1}{}^{-1}e^{\alpha_n(u_{n-1}-u_n)}$$

$$\equiv \mu_0{}^{-1}\mu_1{}^{-1}\mu_2{}^{-1} \cdots \mu_{n-1}{}^{-1} \cdot e^{\alpha_1 x}e^{(\alpha_1-\alpha_2)u_1}e^{(\alpha_2-\alpha_3)u_2} \cdots e^{(\alpha_{n-1}-\alpha_n)u_{n-1}}e^{-\alpha_n u_n}$$

$$\equiv \int^{x}\int^{u_1}\int^{u_2} \cdots \int^{u_{n-1}} e^{\alpha_1 x}e^{(\alpha_1-\alpha_2)u_1} \cdots e^{(\alpha_{n-1}-\alpha_n)u_{n-1}}e^{-\alpha_n u_n}(\,)\,du_1du_2 \cdots du_n$$

This last form is analogous to the forms used in integral equation theory, illuminated by the operational manner of derivation. The subject, of course, must be a form in $u_n$.

(9) $\prod_{i=1}^{n}(D-\alpha_i)^{-k_i}$. This form is but a combination of types $(D-\alpha)^{-1}$ and $(D-\alpha)^{-n}$, and of course the first obvious method of interpretation is that which separates it into partial fractions and then interprets each partial fraction from its form. The procedure will be indicated as follows:

$$\prod_{i=1}(D-\alpha_i)^{-k_i} \equiv \sum_{i=1}\left[\sum_{s=1}^{k_i}N_{is}(D-\alpha_i)^{-}\right] \qquad \text{[II §4]}$$

The multiple-integral interpretation is

$$\prod_{i=1}^{n}(D-\alpha_i)^{-k_i} \equiv \prod_{i=1}^{n} e^{\alpha_i x}D^{-k_i}e^{-\alpha_i x} \qquad \text{[Th. I]}$$

$$\equiv \prod_{i=1}^{n}\mu^{-k_i}e^{\alpha_i(x-u_i)}$$

$$\equiv \prod_{i=1}^{n}\mu^{-1}\frac{(x-u_i)^{k_i-1}}{(k_i-1)!}e^{\alpha_i(x-u_i)}$$

$$\prod_{i=0}^{n-1}\mu_i{}^{-1}\prod_{i=1}^{n}\frac{(u_{i-1}-u_i)^{k_i-1}}{(k_i-1)!}e^{\alpha_i(u_{i-1}-u_i)}$$

where the $\mu_i^{-1}$ integrates with respect to $u_{i+1}$ and puts $u_i$ for $u_{i+1}$, etc., until at the end $\mu_0^{-1}$ integrates with respect to $u_1$ and puts $x$ for $u_1$.

(10) $[D^2 + \beta^2]^{-1}$. By turning this into two partial fractions with linear complex denominators and complex numerators we can interpret simply, as follows:

$$\begin{aligned}
[D^2 + \beta^2]^{-1} &\equiv (D + \beta i)^{-1} \cdot (D - \beta i)^{-1} \\
&\equiv \frac{1}{2\beta i}[(D - \beta i)^{-1} - (D + \beta i)^{-1}] \\
&\equiv \frac{1}{2\beta i}[e^{\beta i x} D^{-1} e^{-\beta i x} - e^{-\beta i x} D^{-1} e^{\beta i x}] \\
&\equiv \frac{1}{2\beta i}[\mu^{-1} e^{\beta i (x-u)} - \mu^{-1} e^{-\beta i (x-u)}] \\
&\equiv \frac{1}{\beta}\mu^{-1} \frac{e^{\beta i (x-u)} - e^{-\beta i (x-u)}}{2i} \\
&\equiv \frac{1}{\beta}\mu^{-1} \sin \beta(x - u) \\
&\equiv \frac{1}{\beta}\int^x \sin \beta(x - u)()du
\end{aligned}$$

(11) $[(D - \alpha)^2 + \beta^2]^{-1}$. By Theorem I this can be turned into a form in which we can use the result just above and then complete the interpretation, *viz.*,

$$\begin{aligned}
[(D - \alpha)^2 + \beta^2]^{-1} &\equiv e^{\alpha x}[D^2 + \beta^2]^{-1} e^{-\alpha x} && \text{[Th. I]} \\
&\equiv e^{\alpha x} \cdot \frac{1}{\beta}\mu^{-1} \sin \beta(x - u) \cdot e^{-\alpha u} && (10) \\
&\equiv \frac{1}{\beta}\mu^{-1} e^{\alpha(x-u)} \sin \beta(x - u) \\
&\equiv \frac{1}{\beta}\int^x e^{\alpha(x-u)} \sin \beta(x - u)()du
\end{aligned}$$

(12) $[D^2 + \beta^2]^{-n}$. From (11) can be derived this form, and it can then be simplified by the definition of an iterated operation.

$$[D^2 + \beta^2]^{-n} \equiv \left[\frac{1}{\beta}\mu^{-1} \sin \beta(x - u)\right]^n$$

$$\equiv \frac{1}{\beta^n} \cdot \underbrace{\mu^{-1} \sin \beta(x - u) \cdot \mu^{-1} \sin \beta(x - u) \cdots \mu^{-1} \sin \beta(x - u)}_{n \text{ factors}}$$

$$\equiv \frac{1}{\beta^n}\mu_0^{-1} \sin \beta(x - u_1)\mu_1^{-1} \sin \beta(u_1 - u_2) \cdots \mu_{n-1}^{-1} \sin \beta(u_{n-1} - u_n)$$

$$\equiv \frac{1}{\beta^n}\prod_{i=0}^{n-1} \mu_i^{-1} \sin \beta(u_{i-1} - u_i)$$

$$\equiv \frac{1}{\beta^n}\prod_{i=0}^{n-1} \mu_i^{-1}\prod_{i=1}^{n} \sin \beta(u_{i-1} - u_i)$$

$$\equiv \frac{1}{\beta^n}\int^x\int^{u_1}\int^{u_2} \cdots \int^{u_{n-1}} \sin \beta(x - u_1) \cdots \sin \beta(u_{n-1} - u_n)(\,)du_1 \cdots du_n$$

(13) $[(D - \alpha)^2 + \beta^2]^{-n}$.

$$\equiv e^{\alpha x}[D^2 + \beta^2]^{-n}e^{-\alpha x} \qquad \text{[Th. I]}$$

$$\equiv e^{\alpha x}\frac{1}{\beta^n}\prod_{i=0}^{n-1} \mu_i^{-1}\prod_{i=1}^{n} \sin \beta(u_{i-1} - u_n)e^{-\alpha u_n} \qquad (12)$$

$$\equiv \frac{1}{\beta^n}\prod_{i=0}^{n-1} \mu_i^{-1}\prod_{i=1}^{n} \sin \beta(u_{i-1} - u_n) \cdot e^{\alpha(x-u_n)}$$

$$\equiv \frac{1}{\beta^n}\int^x\int^{u_1} \cdots \int^{u_{n-1}} \sin \beta(x - u_1) \cdots \sin \beta(u_{i-1} - u_n) \cdot e^{\alpha(x-u_n)}(\,)du_1 \cdots du_n$$

(14) For the other two possible forms we shall merely indicate the more simple product and partial fraction forms, leaving to the student the derivation of the classical forms.

$$\prod_{i=1}^{n}[(D - \alpha_i)^2 + \beta_i^2]^{-1} \equiv \prod_{i=1}^{n}(\mu^2 + \beta_i^2)^{-1}e^{\alpha_i(x-u)}$$

$$\equiv \sum_{i=1}^{n} N_i(\mu^2 + \beta_i^2)^{-1}e^{\alpha_i(x-u)}$$

$$\prod_{i=1}^{n}[(D-\alpha_i)^2+\beta_i^2]^{-k_i} \equiv \prod_{i=1}^{n}(\mu^2+\beta_i^2)^{-k_i}e^{\alpha_i(x-u)}$$
$$\equiv \sum_{i=1}^{n}\left[\sum_{s=1}^{k_i}(M_{si}\mu+N_{si})(\mu^2+\beta_i^2)^{-s}e^{\alpha_i(x-u)}\right]$$

(15) This section is sufficient to show clearly that every inverse operator of algebraic form has its classical counterpart, and therefore the student need feel no hesitancy in using such operators with confidence. We can also say here that this is even sufficient for an introduction to the subject of integral equations, for all of the complicated integrals of that subject are included in one or more of the preceding forms. Later, when we come to the operational attack on integral equations, we shall refer to this section. It illuminates the integrals clearly and shows their structure completely.

(16) Examples of the foregoing interpretations are unnecessary just here, for the student will have numerous illustrations under the applications to differential equations. Even there it is not necessary to show or use the classical forms of integrals, for the operational methods are so much more simple and speedy that the classical forms will be dropped from use after the operational forms are known.

**§7. Operations on Zero.**

(1) For the operator $D^{-1}$ the appendage [Appendix III, §60 (6)] comes from the operation on zero. The appendage, therefore, in this text will always be separated as the result of a zero operation. Every function on which an inverse operation is to be performed will be considered as having zero added to it; thus:

$$f(x), \qquad f(x)+0$$

and

$$F^{-1}(D)\cdot f(x), \qquad F^{-1}(D)\cdot f(x)+F^{-1}(D)\cdot 0$$

The first of these $F^{-1}(D)\cdot f(x)$ is considered as calling only for the particular integral; the second, for the appendage in addition; *i.e.*

$$F^{-1}(D)\cdot f(x) = \text{the particular integral}$$
$$F^{-1}(D)\cdot 0 = \text{the appendage}$$

(2) $D^{-n} \cdot 0$. We have already [II §4 (1)] shown the operational definition of the appendage for $D^{-1}$ but shall derive it here by the classical method. Really, no operational method can be used to interpret this operation on zero, since no simpler operational form can be derived for the operator. We shall say, here,

$$D^{-1} \cdot 0 \equiv \int^x 0 \cdot dx + C_{n-1} = 0 + C_{n-1}$$
$$D^{-2} \cdot 0 \equiv \int^x 0 dx + \int^x C_{n-1} dx + C_{n-2} = 0 + C_{n-1}x + C_{n-2}$$
$$\cdots\cdots\cdots\cdots$$
$$D^{-n} \cdot 0 = 0 + C_{n-1}x^{n-1} + C_{n-2}x^{n-2} + \cdots + C_1x^1 + C_0x^0$$
$$\sum_{k=n-1}^{0} C_k x^k$$

(3) $(D - \alpha)^{-n} \cdot 0$. This form can be simplified before interpretation; thus:

$$\begin{aligned}(D - \alpha)^{-n} \cdot 0 &\equiv (D - \alpha)^{-n} \cdot e^{\alpha x} \cdot e^{-\alpha x} \cdot 0 & [e^{\alpha x}e^{-\alpha x} \equiv 1]\\ &\equiv e^{\alpha x}D^{-n}e^{-\alpha x} \cdot 0 & [\mathrm{I}b]\\ &\equiv e^{\alpha x}D^{-n} \cdot 0 & [e^{-\alpha x} \cdot 0 \equiv 0]\end{aligned}$$

Now the interpretation of (2) is used, and we have

$$\begin{aligned}(D - \alpha)^{-n} \cdot 0 &\equiv e^{\alpha x}[D^{-n} \cdot 0]\\ &= e^{\alpha x}\left[\sum_{k=n-1}^{0} C_k x^k\right]\end{aligned}$$

(4) $\prod_{i=1}^{m} (D - \alpha_i)^{-1} \cdot 0$. By separation into partial fractions we have

$$\prod_{i=1}^{m} (D - \alpha_i)^{-1} \cdot 0 \equiv \sum_{i=1}^{m} N_i(D - \alpha_i)^{-1} \cdot 0$$

Now, by Theorem I$b$, as in (3), we obtain

$$\sum_{i=1}^{m} N_i(D - \alpha_i)^{-1} \cdot 0 = \sum^{m} N_i e^{\alpha_i x} C_i$$

Here the coefficient $N_i$ can be absorbed into the arbitrary constant $C_i$, and we have

$$\prod_{i=1}^{m} (D - \alpha_i)^{-1} \cdot 0 = \sum_{i=1}^{m} C_i e^{\alpha_i x}$$

(5) $\prod_{i=1}^{m} (D - \alpha_i)^{-k_i} \cdot 0$. By separation into partial fractions we obtain the form

$$\prod_{i=1}^{m} (D - \alpha_i)^{-k_i} \cdot 0 \equiv \sum_{i=1}^{m} \left[ \sum_{s=1}^{k_i} N_{is}(D - \alpha_i)^{-s} \cdot 0 \right]$$

Then the bracket falls under (3) and gives

$$\sum_{s=1}^{k_i} N_{is}(D - \alpha_i)^{-s} \cdot 0 = \sum_{s=0}^{k_i-1} e^{\alpha_i x} C_{is} x^s$$

where the $N_{is}$ are absorbed in the $C_{is}$. We thus have

$$\prod_{i=1}^{m} (D - \alpha_i)^{-k_i} \cdot 0 = \sum_{i=1}^{m} \left[ e^{\alpha_i x} \sum_{s=0}^{k_i-1} C_{is} x^s \right]$$

(6) $[D^2 + \beta^2]^{-1} \cdot 0$. Since $D^2 + \beta^2 \equiv (D - i\beta)(D + i\beta)$, we shall have

$$[D^2 + \beta^2]^{-1} \cdot 0 \equiv N_1(D - i\beta)^{-1} \cdot 0 + N_2(D + i\beta)^{-1} \cdot 0$$

and by the same method as above

$$\begin{aligned} &\equiv e^{i\beta x} N_1 D^{-1} \cdot 0 + e^{-i\beta x} N_2 D^{-1} \cdot 0 \\ &= e^{i\beta x} K_1 + e^{-i\beta x} K_2 \end{aligned}$$

Another well-known form for this result is

$$= C_1 \cos \beta x + C_2 \sin \beta x$$

obtained from the first by the substitutions

$$e^{\pm i\beta x} \equiv \cos \beta x \pm i \sin \beta x \quad \text{[Appendix I, §56 (26)]}$$

[This is the complete solution of the differential equation $\dfrac{d^2y}{dx^2} + \beta^2 y = 0.$]

(7) $[(D-\alpha)^2+\beta^2]^{-1}\cdot 0$. This form is immediately transformed into one of the preceding type and thus interpreted:

$$[(D-\alpha)^2+\beta^2]^{-1}\cdot 0 \equiv e^{\alpha x}(D^2+\beta^2)^{-1}\cdot 0 \qquad [\mathrm{I}b]$$
$$= e^{\alpha x}[C_1\cos\beta x + C_2\sin\beta x] \qquad [(6)]$$

(8) $[D^2+\beta^2]^{-n}\cdot 0$. Separate into partial fractions.

$$\begin{aligned}[D^2+\beta^2]^{-n}\cdot 0 &\equiv [(D-i\beta)(D+i\beta)]^{-n}\cdot 0\\
&\equiv \sum_{s=1}^{n} N_{1s}(D-i\beta)^{-s}\cdot 0 + \sum_{s=1}^{n} N_{2s}(D+i\beta)^{-s}\cdot 0\\
&\equiv \sum_{s=1}^{n} N_{1s}e^{i\beta x}D^{-s}\cdot 0 + \sum_{s=1}^{n} N_{2s}e^{-i\beta x}D^{-s}\cdot 0\\
&= \sum_{s=1}^{n} e^{i\beta x}\sum_{h=0}^{s-1} C_{1h}x^h + \sum_{s=1}^{n} e^{-i\beta x}\sum_{h=0}^{s-1} C_{2h}x^h\\
&\quad e^{i\beta x}\sum_{s=1}^{n}\sum_{h=0}^{s-1} C_{1h}x^h + e^{-i\beta x}\sum_{s=1}^{n}\sum_{h=0}^{s-1} C_{2h}x^h\end{aligned}$$

Using only $s = n$ in the double summation, for all the others are included in it, we have

$$= e^{i\beta x}\sum_{h=0}^{n-1} C_{1h}x^h + e^{-i\beta x}\sum_{h=0}^{n-1} C_{2h}x^h$$

We may now substitute for the exponentials

$$e^{i\beta x} = \cos\beta x + i\sin\beta x$$
$$e^{-i\beta x} = \cos\beta x - i\sin\beta x$$

and obtain by collecting terms in sine and cosine the form

$$\left[\sum_{h=0}^{n-1} K_{1h}x^h\right]\cos\beta x + \left[\sum_{h=0}^{n-1} K_{2h}x^h\right]\sin\beta x$$

The dropping of $s = n-1, n-2 \cdots 2, 1$ may easily be shown by letting, say, $n = 3$.

$$\begin{aligned}\frac{}{(D^2+\beta^2)^3} &\equiv \frac{1}{(D+i\beta)^3(D-i\beta)^3}\\
&\frac{N_{11}}{D+i\beta} + \frac{N_{12}}{(D+i\beta)^2} + \frac{N_1}{(D+i\beta)^3}\\
&\qquad + \frac{N_{21}}{D-i\beta} + \frac{N_{22}}{(D-i\beta)^2} + \frac{N_{23}}{(D-i\beta)^3}\end{aligned}$$

If the operator is used on zero, we shall have, after absorbing the $N_{ij}$ with the zeros,

$$
\begin{aligned}
\frac{1}{(D^2+\beta^2)^3}\cdot 0 &\equiv \frac{1}{D+i\beta}\cdot 0 + \frac{1}{(D+i\beta)^2}\cdot 0 + \frac{1}{(D+i\beta)^3}\cdot 0 \\
&\qquad + \frac{1}{D-i\beta}\cdot 0 + \frac{1}{(D-i\beta)^2}\cdot 0 + \frac{1}{(D-i\beta)^3}\cdot 0 \\
&= e^{-i\beta x}\frac{1}{D}0 + e^{-i\beta x}\frac{1}{D^2}0 + e^{-i\beta x}\frac{1}{D^3}0 \\
&\qquad + e^{i\beta x}\frac{1}{D}0 + e^{i\beta x}\frac{1}{D^2}0 + e^{i\beta x}\frac{1}{D^3}0 \\
&= e^{-i\beta x}C_{10} + e^{-i\beta x}(C_{20} + C_{21}x) \\
&\qquad + e^{-i\beta x}(C_{30} + C_{31}x + C_{32}x^2) \\
&\quad + e^{i\beta x}D_{10} + e^{i\beta x}(D_{20} + D_{21}x) \\
&\qquad + e^{i\beta x}(D_{30} + D_{31}x + D_{32}x^2)
\end{aligned}
$$

Now, it is seen that the first two forms in the two rows are included in the third terms, respectively, so that

$$
\begin{aligned}
\frac{1}{(D^2+\beta^2)^3}\cdot 0 &\equiv \frac{1}{(D+i\beta)^3}\cdot 0 + \frac{1}{(D-i\beta)^3}\cdot 0 \\
&= e^{-i\beta x}(C_{10} + C_{11}x + C_{12}x^2) \\
&\qquad + e^{i\beta x}(C_{20} + C_{21}x + C_{22}x^2)
\end{aligned}
$$

In general, it is seen that

$$
\begin{aligned}
\frac{1}{(D^2+\beta^2)^n}\cdot 0 &= e^{-i\beta x}\sum_{h=0}^{n-1} C_{1h}x^h + e^{i\beta x}\sum_{h=0}^{n-1} C_{2h}x^h \\
&\left[\sum_{s=0}^{n-1} K_{1s}x^s\right]\cos\beta x + \left[\sum_{s=0}^{n-1} K_{2s}x^s\right]\sin\beta x
\end{aligned}
$$

(9) $[(D-\alpha)^2+\beta^2]^{-n}\cdot 0$. The only difference between this and the preceding in (8) will be a multiplier $e^{\alpha x}$, as the student can easily find.

$$
\begin{aligned}
[(D-\alpha)^2+\beta^2]^{-n}\cdot 0 &\equiv e^{\alpha x}(D^2+\beta^2)^{-n}\cdot 0 \qquad [Ib,\ C\cdot 0 = 0] \\
&= e^{\alpha x}\left[e^{i\beta x}\sum^{n-1} C_{1s}x^s + e^{-i\beta x}\sum^{n-1} C_{2s}x^s\right] \\
&= e^{\alpha x}\left[\left(\sum_{s=0}^{n-1} K_{1s}x^s\right)\cos\beta x \right. \\
&\qquad \left. + \left(\sum_{s=0}^{n-1} K_{2s}x^s\right)\sin\beta x\right]
\end{aligned}
$$

(10) $\prod_{k=1}^{m} (D^2 + \beta_k^2)^{-1} \cdot 0$. By partial fractions,

$$\equiv \sum_{k=1}^{m} N_k[(D - i\beta)^{-1} \cdot 0 + (D + i\beta)^{-1} \cdot 0]$$

$$\equiv \sum_{k=1}^{m} N_k[e^{i\beta x}D^{-1} \cdot 0 + e^{-i\beta x}D^{-1} \cdot 0]$$

$$= \sum_{k=1}^{m} [C_{1k} \cos \beta x + C_{2k} \sin \beta x]$$

(11) $\prod_{k=1}^{m} [(D - \alpha_k)^2 + \beta_k^2]^{-1} \cdot 0$. First, a somewhat different procedure here will simplify. Separate $(D - \alpha_k)^2 + \beta_k^2$ into linear complex factors, then into partial fractions.

$$\prod_{k=1}^{m} [(D - \alpha_k)^2 + \beta_k^2]^{-1} \cdot 0$$

$$\equiv \prod_{k=1}^{m} [(D - \alpha_k - i\beta_k)(D - \alpha_k + i\beta_k)]^{-1} \cdot 0$$

$$\equiv \sum_{k=1}^{m} [N_{1k}(D - \alpha_k - i\beta_k)^{-1} \cdot 0 + N_{2k}(D - \alpha_k + i\beta_k)^{-1} \cdot 0]$$

$$\equiv \sum_{k=1}^{m} e^{\alpha_k x}[N_{1k}(D - i\beta_k)^{-1} \cdot 0 + N_{2k}(D + i\beta_k)^{-1} \cdot 0]$$

$$\equiv \sum_{k=1}^{m} e^{\alpha_k x}[N_{1k}e^{i\beta_k x}D^{-1} \cdot 0 + N_{2k}e^{-i\beta_k x}D^{-1} \cdot 0]$$

$$\equiv \sum_{k=1}^{m} e^{\alpha_k x}[C_{1k} \cos \beta_k x + C_{2k} \sin \beta_k x]$$

(12) $\prod_{k=1}^{m} [D^2 + \beta_k^2]^{-h_k} \cdot 0$. Partial-fraction separation (Th. I*b*, $N_{ik} \cdot 0 = 0$) and simple interpretation account for this and the form in (13). The results only are given. The student should work these as exercises.

$$\prod_{k=1}^{m} [D^2 + \beta_k^2]^{-h_k} \cdot 0 = \sum_{k=1}^{m} \left[ \left( \sum_{s=0}^{h_k - 1} A_{ks} x^s \right) \cos \beta_k x + \left( \sum_{s=0}^{h_k - 1} B_{ks} x^s \right) \sin \beta_k x \right]$$

(13) $$\prod_{k=1}^{m} [(D - \alpha_k)^2 + \beta_k^2]^{-h_k} \cdot 0 = \sum_{k=1}^{m} e^{\alpha_k x} \left[ \left( \sum_{s=0}^{h_k - 1} A_{ks} x^s \right) \cos \beta_k x + \left( \sum_{s=0}^{h_k - 1} B_{ks} x^s \right) \sin \beta_k x \right]$$

(14) *Examples:*

(a) $\dfrac{1}{D^5} \cdot 0$ (b) $\dfrac{1}{(D-2)^3} \cdot 0$ (c) $\dfrac{1}{(D+1)^4} \cdot 0$

(d) $\dfrac{1}{(D-2)(D+3)} \cdot 0$ (e) $\dfrac{1}{(D+1)(D-3)(D-5)} \cdot 0$

(f) $\dfrac{1}{(D+1)^2(D-4)^5} \cdot 0$ (g) $\dfrac{1}{(D-2)(D-3)^3(D+1)^4} \cdot 0$

(h) $\dfrac{1}{D^2 + \omega^2} \cdot 0$ (i) $\dfrac{1}{D^2 + 2^2} \cdot 0$ (j) $\dfrac{1}{(D-1)^2 + 2^2} \cdot 0$

(k) $\dfrac{1}{D^2 + 2D + 5} \cdot 0$ (l) $\dfrac{1}{D^2 - 3D + 12} \cdot 0$

(m) $\dfrac{1}{(D^2 + \omega^2)^2} \cdot 0$ (n) $\dfrac{1}{(D^2 + 2^2)^3} \cdot 0$

(o) $\dfrac{1}{[(D+3)^2 + 3^2]^2} \cdot 0$ (p) $\dfrac{1}{(D^2 + 2D + 5)^3} \cdot 0$

(q) $\dfrac{1}{(D^2 + 2^2)(D^2 + 3^2)} \cdot 0$

(r) $\dfrac{1}{[(D-1)^2 + 2^2][(D+1)^2 + 3^2]} \cdot 0$

(s) $\dfrac{1}{(D^2 - 3D + 12)(D^2 + 2D + 5)} \cdot 0$

(t) $\dfrac{1}{(D^2 + 2^2)^2(D^2 + 3^2)} \cdot 0$

(u) $\dfrac{1}{(D^2 + 4D + 8)^2(D^2 - 2D + 5)^3} \cdot 0$

(15) This section completely illustrates the operational method of obtaining the complementary functions for ordinary linear differential equations with constant coefficients; and when that subject is reached, reference will be made to this section.

**§8. Inverse Operations on Unity.**

(1) Inverse operations on unity are all simplified by the definition of the inverse and the fundamental theorems I*c* and I*b* [II §5 (5)].

(2) With $n$ as a positive integer

$$D^{-n} \cdot 1 = \frac{x^n}{n!} \qquad \text{[II §4 (4)]}$$

There is nothing mysterious about this. It is merely the result of the interpretation from calculus of the inverse as integration. It has nothing to do with the so-called "unit function" of Heaviside. Unity is as much a subject of the operator as any function might be, and it does not appear in the result.

(3)
$$\begin{aligned}(D - \alpha)^{-n} \cdot 1 &= (D - \alpha)^{-n} \cdot e^{0x} && [e^{0x} \equiv 1]\\ &= e^{0x} \cdot (0 - \alpha)^{-n} \cdot 1 && [\mathrm{I}c]\\ &= (-\alpha)^{-n}\end{aligned}$$

(4) By the same two theorems all the forms below may be easily found by the student.

$$\prod_{i=1}^{m} (D - \alpha_i)^{-1} \cdot 1 = \prod_{i=1}^{m} (-\alpha_i)^{-1}$$

(5)
$$\prod_{i=1}^{m} (D - \alpha_i)^{-k_i} \cdot 1 = \prod_{i=1}^{m} (-\alpha_i)^{-k_i}$$

(6)
$$(D^2 + \beta^2)^{-n} \cdot 1 = \beta^{-2n}$$

(7)
$$[(D - \alpha)^2 + \beta^2]^{-n} \cdot 1 = [(-\alpha)^2 + \beta^2]^{-n}$$

(8)
$$\prod_{i=1}^{m} [(D - \alpha_i)^2 + \beta_i^2]^{-1} \cdot 1 = \prod_{i=1}^{m} [\alpha_i^2 + \beta_i^2]^{-1}$$

(9)
$$\prod_{i=1}^{m} [(D - \alpha_i)^2 + \beta_i^2]^{-k_i} \cdot 1 = \prod_{i=1}^{m} [\alpha_i^2 + \beta_i^2]^{-k_i}$$

(10) But, in general, when combinations of the foregoing are found, it may be necessary to use them consecutively. This will be true when (2) and any of the others are found together. Then the following method is employed:

$$F(D) \cdot 1 = F(0), \qquad \text{provided} \qquad F(0) \neq \infty$$

In the exception, we shall have

$$\begin{aligned} F(D) \cdot 1 &\equiv \frac{1}{D^k \cdot G(D)} \cdot 1 \\ &\equiv \frac{1}{D^k}\left[\frac{1}{G(D)} \cdot 1\right] \\ &= \frac{1}{D^k} \cdot \frac{1}{G(0)} && [\text{I}c;\ G(0) \neq 0] \\ &= \frac{1}{G(0)} \cdot \frac{1}{D^k} 1 \\ &= \frac{1}{G(0)} \cdot \frac{x^k}{k!} && [\S 3\ (2)] \end{aligned}$$

(11) *Examples:*

(a) $\dfrac{1}{D^5} \cdot 1$

(b) $\dfrac{1}{(D-2)^2} \cdot 1$

(c) $\dfrac{1}{(D-1)(D-3)} \cdot 1$

(d) $\dfrac{1}{D^2+5}\ 1$

(e) $\dfrac{1}{D^2+\omega^2} \cdot 1$

(f) $\dfrac{1}{(D-3)^2(D+2)^3} \cdot 1$

(g) $\dfrac{1}{(D-2)^2+7} \cdot 1$

(h) $\dfrac{1}{(D^2+D+2)(D-1)} \cdot 1$

(i) $\dfrac{1}{(D+3)^3(D^2+D+5)} \cdot 1$

(j) $\dfrac{1}{D^2(D+3)^2} \cdot 1$

(k) $\dfrac{1}{(D-1)(D+2)D^3} \cdot 1$

(l) $(D-3)^3 \cdot 1$

(m) $\dfrac{1}{R+LD} \cdot 1$

(n) $\dfrac{1}{D+\dfrac{P}{EI}} \cdot 1$

(o) $(D^2+\omega^2)^2 \cdot 1$

## §9. Operations on Particular Functions.

Here we shall merely give in tabular form the various forms of particular functions and the theorems to be used in each case when the operator is an inverse.

| | $F^{-1}(D) \cdot f(x)$, where | Use |
|---|---|---|
| (1) | $f(x) = k$ (a constant) | $k = k \cdot 1$ and carry $k$ across the operator. Then use the theorems of §8 |
| (2) | $f(x) = e^{ax}$ | Theorem I*c*, or Theorems I*b*, and §8 |
| (3) | $f(x) = x^m$ | Direct integration; or expand the operator by §4(12) and following |
| (4) | $f(x) = \left.\begin{matrix}\sin\\ \cos\\ \sinh\\ \cosh\end{matrix}\right] ax$ | Theorems III or IV; exception, use Theorem I and then III, IV |
| (5) | $f(x) = e^{ax}\phi(x)$ | Theorem I*b*, then according to the form of $\phi(x)$ |
| (6) | $f(x) = x^n\phi(x)$ | Theorem VI |
| (7) | $f(x) = e^{ax}x^m\phi(x)$ | Theorems I*b* and VI |

(8) *Examples:*

(a) $\dfrac{1}{EID^2}(-Pc)$

(b) $\dfrac{1}{D^2}(-g)$

(c) $\dfrac{1}{EID^2 + P}(Pa)$

(d) $\dfrac{1}{D^2 + kp}(-g)$

(e) $\dfrac{1}{R + \dfrac{1}{cD}}E_0e^{-\beta x}$

(f) $\dfrac{1}{D - a}e^{bx}$

(g) $\dfrac{1}{D^2 - 3D + 2}e^{x}$

(h) $\dfrac{1}{D^5 - m^2D^3}e^{ax}$

(i) $\dfrac{1}{R + LD}E \cos \omega x$

(j) $\dfrac{1}{LD^2 + RD + \dfrac{1}{c}}E_0\, \omega \cos \omega x$

(k) $\dfrac{1}{D^2 + 2kD + k^2 + \lambda^2}A \cos px$

(l) $\dfrac{1}{D^2 + k^2} k \cos px$

(m) $\dfrac{1}{D^2 - \omega^2} \sin \omega x$

(n) $\dfrac{1}{D^2 + \omega^2} \sin \omega x$

(o) $\dfrac{1}{D^2 - 2D + 1} x^2 e^{3x}$

(p) $\dfrac{1}{(D - 2)^3} x^2 e^{2x}$

(q) $\dfrac{1}{(D^4 - 1)^2} x^4 e^x$

(r) $\dfrac{1}{D + 1} x^2 \cos x$

(s) $\dfrac{1}{D^2 + p^2} e^{ax} \sin px$

(t) $\dfrac{1}{D^2 - 4} x^3$

(u) $\dfrac{1}{D^2 + p^2} f(x)$

(v) $\dfrac{1}{D + 1} x^7$

(w) $\dfrac{1}{D^2 - 2aD + b^2} e^{-px} \sin qx$

(y) $(D - a)^{4 \cdot n} e^{ax} \sin px$

# CHAPTER III

## APPLICATIONS TO ORDINARY LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

### §10. Linear Differential Equations.

(1) It has often been said that a large majority of the differential equations of mathematical physics are of the type of ordinary linear differential equations with constant coefficients. Be that as it may, it can certainly be shown that in certain fields, electrical circuits, mechanical vibrations, etc., we do have largely this type in use. Before proceeding to the general theory of the application to this type, let us look at a few illustrative examples.

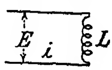
(2) *Electrical.* The common symbols for parts of electric circuits, together with their diagrammatic representation, are as follows:

| | | |
|---|---|---|
| Time, $t$, | | $O$, $E_0$ |
| Current, $I$, $i$, | | $E_0\, e^{-\beta t}$ |
| Electromotive force, $E$, $e$. . . . . . . . . . . . . . . . . . . . . | | $E_0 \cos \omega t$ |
| Resistor, $R$, | $Ri.$ | $E_0\, e^{-\beta t} \cos \omega t$ |
| Inductor, $L$, | $L\dfrac{di}{dt}$ | $f(t)$ |
| Condensor or capacitor, $C$, | $\dfrac{1}{C}\displaystyle\int idt$ | |

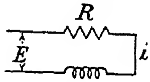
The differential equation for an electric circuit can always be written down in terms of the foregoing and is one with constant coefficients. The electromotive force (the action) is equated to the sum of the resisting forces (the reaction).

(a) $E = Ri$, ohm's law

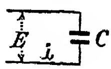

(b) $E = L\dfrac{di}{dt}$



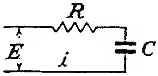

(c) $E = Ri + L\dfrac{di}{dt}$



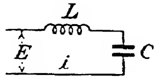

(d) $E = \dfrac{1}{C}\displaystyle\int idt$



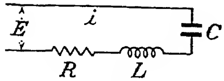
(e) $E = Ri + \frac{1}{C}\int i\,dt$




(f) $E = L\frac{di}{dt} + \frac{1}{C}\int i\,dt$




(g) $E = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt$




Equations (*d*), (*f*), and (*g*) are integro-differential and by differentiation will give differential equations, *e.g.*:

$$(d') \qquad \frac{dE}{dt} = \frac{i}{C}$$

$$(f') \qquad \frac{dE}{dt} = L\frac{d^2i}{dt^2} + \frac{i}{C}$$

$$(g') \qquad \frac{dE}{dt} = R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C}$$

(*h*) When $E = E_0 \cos \omega t$, $R = 0$, and $\frac{1}{C} = \omega^2 L$, we have the resonance circuit represented by

$$E_0\,\omega \sin \omega t = L\frac{d^2i}{dt^2} + \omega^2 Li$$

(*i*) When $E = E_0\, e^{-\beta t} \cos \omega t$, $R = 2\beta L$, and $\frac{1}{C} = (\omega^2 + \beta^2)L$, we have the damped resonance circuit

$$E_0\sqrt{\omega^2 + \beta^2}\, e^{-\beta t} \sin (\omega t + \alpha) = L\frac{d^2i}{dt^2} + 2\beta L\frac{di}{dt} + (\omega^2 + \beta^2)Li$$

(3) *The Pendulum.* *a.* The simple bob pendulum in a vacuum has the differential equation

$$l\,\frac{d^2\theta}{dt^2} = -g \sin \theta$$

which for small oscillations ($\sin \theta \approx \theta$) becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

*b*. If the resistance of the medium is taken into account, a damping factor is present in the equation

$$\frac{d^2\theta}{dt^2} = 2k\frac{d\theta}{dt} - \frac{g}{l}\theta$$

Here the energy, restoring force, and damping are in equilibrium.

*c*. The bifilar suspension pendulum, with small free oscillations, has the equation

$$mk^2\frac{d^2\theta}{dt^2} = -mg\frac{b^2}{l}\theta$$

where $l$ is the suspension length, $g$ is gravity, $m$ is the total mass of the bar, $2b$ is the length between the suspension filaments, and $k$ is the radius of gyration of the bar.

*d*. The horizontal seismograph has the following equation:

$$\frac{d^2x}{dt^2} + \frac{g}{l}x \quad \frac{g}{l}C\cos\omega t$$

(4) *Oscillations*. *a*. All vibrating systems about a position of equilibrium where the resistance of the medium is neglected have the differential equation of simple harmonic motion, the type form being

$$\frac{d^2x}{dt^2} + n^2x = 0 \qquad \text{[Lamb]}$$

*b*. A stretched string of length $L$, having tension $T$ and a particle of mass $m$ in the middle, set into vibration by pulling out laterally gives

$$m\frac{d^2x}{dt^2} = -4T\cdot\frac{x}{l} \qquad \text{[Lamb]}$$

*c*. A ship oscillating about a longitudinal axis through the mass center gives

$$k^2\frac{d^2\theta}{dt^2} = -gC\theta \qquad \text{[Lamb]}$$

where $C$ is the meta-centric height, $k$ is the radius of gyration, and $g$ is gravity.

*d*. Central force varying as distance, damped:

$$\frac{d^2y}{dt^2} + \left(\mu - \frac{1}{4}k^2\right)y = 0 \qquad \text{[Lamb]}$$

This has three cases:

$$\mu \gtreqless \tfrac{1}{4}k^2, \qquad i.e., \qquad n^2 = \mu - \tfrac{1}{4}k^2 \gtreqless 0$$

*e*. Lateral displacement $y$ of a portion of a thin vertical shaft in rapid revolution; $x$ being the vertical height of the portion from the lower support:

$$\frac{d^4y}{dt^4} - n^4y = 0 \qquad \text{[Piaggio]}$$

*f*. Oscillations with increasing amplitude:

$$\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + 2y \qquad k$$

*g*. Forced oscillations, damping force proportional to velocity, restoring force, give Helmholtz's equation

$$\frac{d^2y}{dt^2} + 2a\frac{dy}{dt} + n^2y = b \cos \omega t$$

(5) *Beams*. In work on strength of materials appear certain bending-moment equations, the integration of which brings into the picture the shears and elastic curves. We have here quite a variety of differential equations, depending upon the kind of supports and the loading. With $E$ for the modulus of elasticity of the material of the beam, $W$ for an isolated load, $I$ for the moment of inertia of a section, $l$ for length, $x$ for the distance of the section from the origin, $\omega$ for uniform unit load, and $y$ for the ordinate of the elastic curve, we have

(*a*) $EI\frac{d^2y}{dt^2} = -\frac{1}{2}Wx$

(*b*) $= -\tfrac{1}{2}\omega x(l - x)$

(*c*) $= \frac{\omega}{2}(l - x)^2$

(*d*) $= -Wl + (W + \omega l)x - \frac{\omega}{2}(l^2 + x^2)$

(*e*) $= \frac{\omega}{2}\left(\frac{l^2}{4} - x^2\right)$

(*f*) $= -\frac{\omega}{2}x^2$

(6) *Miscellaneous.* *a.* Distribution of pressure in an atmosphere of uniform temperature: $z$, vertical distance; $p$, pressure; $\rho$, density; $H = \frac{p_0}{\rho_0}$.

$$\frac{dp}{dz} \qquad -\frac{p}{H} \qquad \text{[Lamb]}$$

*b.* Pressure interior to a liquid globe due to the mutual attraction of the parts of the liquid:

$$\frac{dp}{dr} = -g\rho\frac{r}{a} \qquad \text{[Lamb]}$$

where $g$ = gravity at surface (force per unit mass).
$\rho$ = density.
$r$ = distance from center.
$a$ = radius of sphere.

*c.* Increase in length of a bar hanging vertically and stretched by its own weight:

$$E\frac{dz}{dx} = \rho(l - x) \qquad \text{[Lamb]}$$

*d.* Fourier's equation for the propagation of heat in a cylindrical bar

$$\frac{d^2V}{dx^2} - \beta^2V = 0 \qquad \text{[Mellor]}$$

**§11. The Single General Equation.**

(1) Every *linear differential equation* with constant coefficients can be written in the form

$$F(D) \cdot y = f(x)$$

where $D \equiv \frac{d}{dx}$, and $F(D) \equiv \sum_k a_kD^k$, the $a_k$ being constants, real or complex.

The largest $k$ is the order of the equation, though this class may include as special cases those in which $k = \infty$. In the latter cases, the $F(D)$ may be a finite form in $D$, capable of being expanded into an infinite series in $D$, convergent or not. The $f(x)$ may be either a finite form or an infinite series.

(2) The operational solution requires the addition of zero to $f(x)$; thus,

$$F(D) \cdot y = f(x) + 0$$

Then the complete solution for $y$ will be obtained by operating on the left by $F^{-1}(D)$.

$$F^{-1}(D) \cdot F(D) \cdot y = F^{-1}(D) \cdot f(x) + F^{-1}(D) \cdot 0$$

*i.e.*, since $F^{-1} \cdot F \equiv 1$

$$y = F^{-1}(D) \cdot f(x) + F^{-1}(D) \cdot 0$$

The first term on the right gives the *particular integral;* and the second, the *complementary function.* The latter will be treated first.

(3) *The Complementary Function.* It is obvious at first sight, in the light of Chap. II (§7, "Operations on Zero") that the differential equations may immediately be solved for the complementary function. Turn the differential equation into the operational form, and solve algebraically for the dependent variable, which will give an inverse operation on zero.

(4) *Examples.* Solve the following for the complementary function:

(*a*) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 0$

(*b*) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = 0$

(*c*) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0$

(*d*) $2\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 2y = 0$

(*e*) $9\dfrac{d^2y}{dx^2} + 18\dfrac{dy}{dx} - 16y = 0$

(*f*) $a\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx}$

(*g*) $\dfrac{d^2r}{dx^2} - a^2r = 0$

(*h*) $\dfrac{d^2y}{dx^2} + y = 0$

(*i*) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$

(*j*) $3\dfrac{d^2y}{dx^2} - 10\dfrac{dy}{dx} + 3y = 0$

(*k*) $\dfrac{d^2y}{dx^2} - 2\lambda\dfrac{dy}{dx} + (\lambda^2 + \mu^2)y = 0$

(*l*) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 13y = 0$

(*m*) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 2y = 0$

(*n*) $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = 0$

(*o*) $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 2y = 0$

(*p*) $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} = 0$

(*q*) $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + y = 0$

(*r*) $\dfrac{d^3y}{dx^3} - 8y = 0$

(*s*) $\dfrac{d^4y}{dx^4} - 3\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 0$

(*t*) $\dfrac{d^4y}{dx^4} + a^4y = 0$

(u) $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0$

(v) $\frac{d^4y}{dx^4} - y = 0$ (w) $4\frac{d^5y}{dx^5} - 3\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = 0$

(x) $\frac{d^6y}{dx^6} - 2\frac{d^5y}{dx^5} + 3\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

(y) $\frac{d^8y}{dx^8} = y$

(5) *The Particular Integral.* For this also, the necessary procedure has already been presented so that reference need be made only to Chap. II (§8 and §9, "Operations on Unity and on Particular Functions of $x$"). The student should therefore be able to solve the following for particular integrals as well as complementary functions and hence obtain the general solutions.

(6) *Examples:*

(a) $(D - 1) \cdot y = e^{2x}$
(b) $(D^2 + D) \cdot y = e^x$
(c) $(D^2 - 3D + 2) \cdot y = e^x$
(d) $(D^2 - 3D + 2) \cdot y = e^{3x}$
(e) $(D^2 - 5D + 6) \cdot y = e^{2x}$
(f) $(D^2 - a^2) \cdot y = e^{ax} + e^{nx}$
(g) $(D^2 + 9) \cdot y = 60 \cdot e^{-t}$
(h) $(D - a)^a \cdot y = a^x + e^{bx}$
(i) $(D^3 \pm 4D) \cdot y = x^2 - 3e^{2x}$
(j) $(D - 1) \cdot y = \sin x$
(k) $D^2 \cdot y = \cos x$
(l) $(D^2 - 5D + 6) \cdot y = 100 \cdot \sin 4x$
(m) $(D^2 + 3) \cdot y = \sin x + \frac{1}{3} \sin 3x$
(n) $(D^2 + b^2) \cdot y = E \cdot \sin pt$
(o) $(D^2 - 1) \cdot y = \cosh x$
(p) $(D - 1) \cdot y = (x + 3)e^{2x}$
(q) $(D^2 + 2D + 1) \cdot y = x^2e^{3x}$
(r) $(D - 1)^2 \cdot y = -e^x \log (1 - x)$
(s) $(D + 1)^3 \cdot y = xe^{-x}$
(t) $(D^3 + 1) \cdot y = xe^{2x}$
(u) $(D^2 + n^2) \cdot y = x^2 \cos ax$
(v) $(D^2 - 4D + 3) \cdot y = e^{2x} \sin x$
(w) $(D^3 - 2D + 4) \cdot y = e^x \cos x$

At the end of this chapter will be found a set of examples taken from engineering and science.

**§12. The Heaviside Expansion Theorem.**

(1) One of the most interesting of practical applications of the operational method is that made by Oliver Heaviside in the form of his "expansion theorem." It has become famous in the electrical field and has been much written about. It is the basis of many extensions, all of which leave something to be desired,

from the standpoint of clarity, of exegesis, or of completeness. Here will be given an elementary derivation of the theorem, showing what its real structure is.

(2) Under partial fraction theory we have,

if

$$\phi(x) = (x - a)\psi(x)$$

$$\frac{f(x)}{\phi(x)} = \frac{f(x)}{(x - a)\psi(x)} \equiv \frac{A}{x - a} + \frac{g(x)}{\psi(x)}$$

then

$$\frac{f(x)}{\psi(x)} \equiv A + (x - a)\frac{g(x)}{\psi(x)}$$

For $x = a$,

$$\frac{f(a)}{\psi(a)} = A$$

and since

$$\phi(x) \equiv (x - a) \cdot \psi(x)$$

by differentiation,

$$\phi'(x) \equiv (x - a) \cdot \psi'(x) + \psi(x)$$

which, for $x = a$, gives

$$\phi'(a) \equiv \psi(a) \equiv \frac{d\phi(x)}{dx}$$

so that we have

$$A \equiv \frac{f(a)}{\phi'(a)}$$

For all factors of $\phi(x)$ distinct and linear, we should have

$$\frac{f(x)}{\phi(x)} \equiv \sum_{i=1}^{n} \frac{f(a_i)}{\phi'(a_i)} \cdot \frac{1}{x - a_i}$$

By analogy,

$$\left[\frac{\phi(D)}{f(D)}\right] \cdot y = K \qquad \text{[}K\text{ a constant]}$$

or

$$y = \frac{f(D)}{\phi(D)} \cdot K \qquad \text{[the particular integral]}$$

$$= \sum_{i=1}^{n} \frac{f(a_i)}{\dfrac{d\phi}{dD}} \quad \frac{}{D - a_i} \cdot K$$

If $K \equiv 1$, we have [in Heaviside's treatment, the 1 is his "unit function"]

$$\frac{1}{D - a_i} \cdot 1 = \frac{1}{-a_i}$$

If $K \equiv 0$, we have

$$\frac{}{D - a_i}\, 0 = C_i e^{a_i t}$$

Thus,

$$y = \frac{f(D)}{\phi(D)}[1 + 0]$$

$$= \sum_{i=1}^{n} \frac{f(a_i)}{\left.\frac{d\phi}{dD}\right|_{D=a_i}} \left[ -\frac{1}{a_i} + C_i e^{a_i t} \right]$$

In dynamical systems, we can safely assume that at the time forces are applied all coordinates are zero; *i.e.*, here

$$y = 0, \qquad \text{when} \qquad t = 0$$

Applying this,

$$0 = \sum_{i=1}^{n} \frac{f(a_i)}{\left.\frac{d\phi}{dD}\right|_{D=a_i}} \left[ -\frac{1}{a_i} + C_i \right]$$

or

$$C_i = \frac{1}{a_i}$$

giving

$$y = -\sum_{i=1} \frac{f(a_i)}{\left.a_i \frac{d\phi}{dD}\right|_{D=a_i}} + \sum_{i=1}^{n} \frac{f(a_i) e^{a_i t}}{a_i \frac{d\phi}{dD}}$$

But

$$\frac{f(a_i)}{\left.{}_{i=1} a_i \frac{d\phi}{dD}\right|_{D=a_i}} \equiv \frac{f(D)}{\phi(D)}$$

and then we have

$$y = -\frac{f(0)}{\phi(0)} + \sum_{i=1} \frac{f(a_i) e^{a}}{a_i \frac{d\phi}{dD}}\;{}_{D=a_i}$$

(3) Now, if we call

$$-\frac{f(0)}{\phi(0)} \equiv \frac{Y(0)}{Z(0)} \qquad \text{and} \qquad \frac{f(a_i)e^{a_it}}{\frac{d\phi}{dD}\Big|_{D=}} \qquad \sum_{p_1,p_2,\ldots} \frac{Y(p)e^{pt}}{p\cdot\frac{dZ}{dp}}$$

we have the clue to the Heaviside expansion theorem. Thus it is obvious that Heaviside's form

$$y \quad \frac{Y(0)}{Z(0)} + \qquad \frac{Y(p)e^{pt}}{p\ \frac{dZ}{dp}}$$

contains the particular integral and the complementary function of our linear differential equation theory, where the initial conditions (two of them) are put in. The zero value of $K$ gives the $\frac{Y(0)}{Z(0)}$, and the unit value of $K$ gives the $\sum_p \frac{Y(p)e^{pt}}{pZ'(p)}$. Also, the $Z$ function contains only linear factors, each one distinct.

(4) Heaviside himself recognized that his theorem left much to be desired, but he did not remedy it. Some recent writers have developed the case where repeated real roots are present. F. D. Murnaghan, by the use of Cauchy's method,* has shown that additional terms must be present for repeated roots. J. R. Carson† developed the expansion theorem from a system of second-order equations in $n$ variables, in which he considered repeated roots. Neither these nor any of the many other interpretations of the theorem have been made clear enough for the ordinary student to use. An extension could easily be written on the basis of the expansion of the operator into partial fractions, taking into consideration all four forms of partial fractions, and this would necessitate four summation sets, one for each of the four types. But nothing practical would be gained, because the foregoing operational method is sufficient and simple enough for all possible practical cases, and it is quite easy to evaluate the arbitrary constants for any terminal conditions found in physical situations. Moreover, one may be certain, when he has the general solution of the underlying differential equation, that he has covered the problem completely.

* *Amer. Math. Monthly*, **23**, (1927), 81–89.

† *Phys. Rev.*, **2**, *X* (1917), 217–225.

(5) The expansion theorem is an interesting case of the insertion of terminal conditions before solution, so that the arbitrary constants of integration never appear. It is a coincidence due possibly to the fact that the right-hand side of the underlying differential equation is a constant.

**§13. Solution by Infinite Series.**

(1) Whenever the operational solution gives us a particular integral which on interpretation is found to be unintegrable, then, as in the classical method, solution by infinite series must be resorted to. This is more easily accomplished by the operational method than by any other.

(2) An illustration will suffice to show the method. Take the differential equation

$$\frac{dy}{dx} - ay = \frac{1}{x}$$

Its operational form is

$$(D - a) \cdot y = \frac{1}{x} + 0$$

The formal solution is

$$y = \frac{1}{D - a} \cdot \frac{1}{x} + \frac{1}{D - a} \cdot 0$$

The complementary function is $Ce^{ax}$, but the particular integral on interpretation is

$$= e^{ax}\int \frac{e^{-ax}}{x} dx$$

which is not integrable in finite form.

(3) The operational method would be to expand $\frac{1}{(D - a)}$ into an infinite series of either ascending or descending powers of $D$ and then to operate term by term on $\frac{1}{x}$; thus

$$\frac{1}{D-a}\cdot\frac{1}{x} \equiv -\frac{1}{a}\cdot\frac{1}{1-\dfrac{D}{a}}\cdot\frac{1}{x}$$

$$\equiv -\frac{1}{a}\left(1+\frac{D}{a}+\frac{D^2}{a^2}+\cdots\right)\cdot\frac{1}{x}$$

$$= -\frac{1}{a}\left(\frac{1}{x}-\frac{1!}{ax^2}+\frac{2!}{a^2x^3}-\frac{3!}{a^3x^4}+-\cdots\right)$$

$$= -\frac{1}{ax}\sum_{k=0}^{\infty}\frac{k!}{(-ax)^k}$$

or

$$\frac{1}{D-a}\cdot\frac{1}{x} \equiv \frac{1}{D}\cdot\frac{1}{1-\dfrac{a}{D}}\cdot\frac{1}{x}.$$

$$\equiv \frac{1}{D}\left(1+\frac{a}{D}+\frac{a^2}{D^2}+\cdots\right)\cdot\frac{1}{x}$$

$$= \frac{1}{D}\left(\frac{1}{x}+a\log x+\cdots\right)$$

a much more difficult form to complete.

(4) Whenever such a type of equation must be solved, it will be found necessary to examine the series solution to see whether or not it satisfies the differential equation under suitable terminal conditions; *i.e.*, if it is convergent in the region in which the differential equation is valid. The series is to be rejected if it is not convergent.

(5) It will be sufficient here to say that it is unlikely that any such equation will come up in engineering work. Most of the equations that the student meets will be regular.

**§14. Examples.**

In the set of problems on pp. 68 and 69 are contained most of those given in §10, and some additional ones taken from practical work. Here $p \equiv \frac{d}{dt}$, $D \equiv \frac{d}{dx}$, and $\vartheta \equiv \frac{d}{dz}$.

| No. | Equations | Terminal conditions | Source |
|---|---|---|---|
| 1 | $(p + 1/190)K = 0$ | . . . . . . . . . . . . . . . | Mellor |
| 2 | $\left(\vartheta + \frac{1}{H}\right)p = 0$ | $\begin{cases} p = p_0 \\ z = 0 \end{cases}$ | Lamb |
| 3 | $\left(p^2 + \frac{1}{LC}\right)i = 0$ | $\begin{cases} i = 0 \\ t = 0 \end{cases}$ $\begin{cases} i = I \\ t = t \end{cases}$ | |
| 4 | $(Ap^2 + MH)y = 0$ | . . . . . . . . . . . . . . . | Emtage |
| 5 | $(p^2 + 2kp + \omega^2)\theta = \omega^2\alpha$ | . . . . . . . . . . . . . . . | Emtage |
| 6 | $\left(p^2 + \frac{R}{L}p + \frac{1}{LC}\right)i = 0$ | | |
| 7 | $(p^2 + k^2)y = 0$ | $\begin{cases} y = 0 \\ t = 0 \end{cases}$ $\begin{cases} y = Y \\ t = \frac{\pi}{2k} \end{cases}$ | |
| 8 | $\left(p^2 + \frac{g}{l}\right)\theta = 0$ | | |
| 9 | $\left(mp^2 + 4\frac{P}{l}\right)x = 0$ | | |
| 10 | $\left(D^2 + \frac{W}{EI}\right)y = 0$ | | |
| 11 | $\frac{d^2T}{d\theta^2} = T$ | $\begin{cases} \theta = 0 \\ T = T_0 \end{cases}$ $\begin{cases} \theta = \alpha \\ T = 0 \end{cases}$ | Lamb |
| 12 | $\left(p^2 + \frac{gb^2}{k^2l}\right)\theta = 0$ | . . . . . . . . . . . . . . . | Lamb |
| 13 | $\left(p^2 + \mu - \frac{k^2}{4}\right)y = 0$ | . . . . . . . . . . . . . . . | Lamb |
| 14 | $p^2 \cdot s = a$ | . . . . . . . . . . . . . . . | Murray |
| 15 | $(D^4 + n^4)y = 0$ | . . . . . . . . . . . . . . . | Piaggio |
| 16 | $Lpi = E$ | | |
| 17 | $(Lp + R)i = E$ | | |
| 18 | $\left(R + \frac{1}{cp}\right)i = E$ | | |
| 19 | $\left(R + Lp + \frac{1}{cp}\right)i = E$ | | |
| 20 | $p^2x = -g$ | $\begin{cases} x = x_0 \\ t = 0 \end{cases}$ $\begin{cases} px = u_0 \\ t = 0 \end{cases}$ | Lamb |
| 21 | $EID^2x = -M$ | . . . . . . . . . . . . . . . | Boyd |
| 22 | $(EIp^2 + P)y = Pa$ | . . . . . . . . . . . . . . . | Campbell |
| 23 | $(p^2 + kp)u = -g$ | . . . . . . . . . . . . . . . | Lamb |
| 24 | $(ED + \rho)x = \rho l$ | . . . . . . . . . . . . . . . | Lamb |
| 25 | $px = k_1(a - x)^2 - k_2x^2$ | . . . . . . . . . . . . . . . | Bateman |
| 26 | $EID^2y = -\frac{1}{2}Wx$ | . . . . . . . . . . . . . . . | Lamb |
| 27 | $(EID^2 - Q)y = -\frac{1}{2}\omega x^2$ | . . . . . . . . . . . . . . . | Murray |

| No. | Equations | Terminal conditions | Source |
|---|---|---|---|
| 28 | $EID^2y = M_0 + \frac{\omega}{2}(lx - x^2)$ | .................. | Boyd |
| 29 | $Lpi = E \sin \omega t$ | | |
| 30 | $(R + Lp)i = E \cos \omega t$ | | |
| 31 | $\left(R + \frac{1}{Cp}\right)i = E \cos \omega t$ | | |
| 32 | $\left(R + Lp + \frac{1}{Cp}\right)i = E \sin \omega t$ | | |
| 33 | $\left(Lp^2 + Rp + \frac{1}{C}\right)Q = E \cos \omega t$ | | |
| 34 | $p^2x = a \cos nt$ | | |
| 35 | $(p^2 + 2kp + b^2)s = a \cos qt$ | | |
| 36 | $(p^2 + a^2)y = k \cos qt$ | | |
| 37 | $(p^2 + n^2)s = k[3 \cos (nt + \epsilon) + \cos 3(nt + \epsilon)]$ | .................. | Lamb |
| 38 | $(Mk^2p^2 + Mgh)\theta = -Xh$ | .................. | Lamb |
| 39 | $(p^2 - \omega^2)r = -g \sin \omega t$ | $\begin{cases} r = 0 \\ t = 0 \end{cases}$ $\begin{cases} pr = 0 \\ t = 0 \end{cases}$ | Cohen |
| 40 | $(D^2 - 2aD + b)y = Ce^{px} \sin (qx + \alpha)$ | .................. | Forsythe |
| 41 | $(p^2 + 2hp + h^2 + q^2)y = ke^{-ht} \cos qt$ | | |

# CHAPTER IV

## ALGEBRAIC THEOREMS

**§15. Need of Algebraic Theorems.**

Much of the difficulty that engineering students encounter in gaining facility in the use of the operations of the calculus lies in their unfamiliarity with many of the most elegant theorems of algebra. This is partly due to the speed with which they are driven over their algebra and partly due to the fact that college algebras do not contain some of the really useful theorems or methods. For instance, the really elegant theorems for the development of partial fractions are not in the algebra texts and in only one elementary calculus text. The most useful method of evaluating numerical determinants is not in any of the algebras in common use in America. Matrix theory is not touched upon at all in any engineering mathematics courses. Since our subject is the algebraization of the calculus, it is necessary for the student to be familiar and facile with those algebraic theorems used, and we are therefore in this chapter including most of the algebraic theorems not covered thoroughly and carefully or possibly not covered at all in the preparation that our readers may have had. Partial fractions have already been covered in Chap. II.

**§16. Determinants.**

(1) *Definition.* In the ordinary sense a determinant is a square array of terms within vertical bars, considered as a single related set. It has the same number of rows as columns, and the notation used is significant of rows and columns. Thus:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

These may be indicated by shortened symbols, respectively,

$$\left| \begin{matrix} ab \\ a_{12} \end{matrix} \right|, \qquad \left| \begin{matrix} abc \\ a_{13} \end{matrix} \right| \qquad \left| \begin{matrix} abcd \\ a_{14} \end{matrix} \right|$$

In general, $| a_{1n} |$ indicates a determinant with $n$ rows and $n$ columns, $n$ being called its order. Also, $| a_{ij} |$, or $\mathbf{a}_{ij}$, or simply $\mathbf{a}$ indicates the same set without giving its order.

(2) The relation among the terms is defined by the theorems that follow. Rigorous proofs or derivations will not be given, though the structure will be shown in some cases. Full details can be found in any good book on algebra, determinants, or theory of equations. Also, only the theorems needed in the following chapters are included.

(3) *Minors.* The *element* in the $i$th row and $j$th column of the determinant $| a_{ij} |$ is indicated by $a_{ij}$. If we cut out the $i$th row and $j$th column and form a determinant of all the elements remaining, we shall have what is called the *complementary* minor of $a_{ij}$; *i.e.*, with

$$| a_{ij} | \equiv \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1j} & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,j-1} & a_{2j} & a_{2,j+1} & \cdots & a_{2n} \\ \cdot\ \cdot & \cdot\ \cdot & \cdots & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdots & \cdot\ \cdot \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i1} & a_{i2} & \cdots & a_{i,j-1} & a_{ij} & a_{i,j+1} & \cdots & a_{in} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \cdot\ \cdot & \cdot\ \cdot & \cdots & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdots & \cdot\ \cdot \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{nj} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

the minor of $a_{ij}$ will be

$$M_{ij} \equiv \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \cdot\ \cdot & \cdot\ \cdot & \cdots & \cdot\ \cdot & \cdot\ \cdot & \cdots & \cdot\ \cdot \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \cdot\ \cdot & \cdot\ \cdot & \cdots & \cdot\ \cdot & \cdot\ \cdot & \cdots & \cdot\ \cdot \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

$M_{ij}$ is also called a *first minor* of $| a_{ij} |$. It is of order $n - 1$. A *second minor* of $| a_{ij} |$ is any such array from it after *any two*

rows and columns have been deleted, and it will be of order $n - 2$. For example, if

$$|a_{ij}| \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

then $\begin{vmatrix} a_{21} & a_{24} \\ a_{31} & a_{34} \end{vmatrix}$ is one of its second minors. Here the first and fourth rows and the second and third columns have been deleted. $\begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix}$ will be another second minor. Each element of the $| a_{ij} |$ will be an $(n - 1)$st minor, because, to obtain it, $n - 1$ rows and columns will be deleted. A *diagonal* minor is any such secondary array whose diagonal row is in the diagonal row of the original determinant. In the last example,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{vmatrix}$$

are diagonal minors.

(4) The *cofactor* of an element is its complementary minor with a sign attached determined by the $i$ and $j$ of the element; thus,

$$A_{ij} = (-1)^{i+j} \cdot M_{ij} \text{ is the cofactor of } a_{ij}$$

(5) In any determinant, if the rows in order are made into columns in order, its value is unchanged. Thus,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \equiv \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

In general, $| a_{ij} | \equiv | a_{ji} |$.

(6) If any two rows (columns) are interchanged, the sign is changed, but not the absolute numerical value. Thus,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{vmatrix} \quad \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{vmatrix}$$

As a consequence of this, if any two rows (columns) are identical, the determinant has the value zero.

$$\Delta = \begin{vmatrix} a_{11} & a_{11} & a_{12} \\ a_{21} & a_{21} & a_{22} \\ a_{31} & a_{31} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{11} & a_{12} \\ a_{21} & a_{21} & a_{22} \\ a_{31} & a_{31} & a_{33} \end{vmatrix} = -\Delta$$

by interchanging first and second columns. Thus,

$$\Delta = -\Delta \qquad \text{or} \qquad 2\Delta = 0$$

giving

$$\Delta = 0$$

As another consequence of this, any row (column) can be moved over one *or more* rows (columns) by changing the sign of the determinant for every such move. Thus,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & a_{13} & a_{12} \\ a_{21} & a_{23} & a_{22} \\ a_{31} & a_{33} & a_{32} \end{vmatrix} \qquad \begin{vmatrix} a_{13} & a_{11} & a_{12} \\ a_{23} & a_{21} & a_{22} \\ a_{33} & a_{31} & a_{32} \end{vmatrix}$$

(7) If every element in any row (column) is multiplied by the same number, the determinant itself has that number as a factor. Thus,

$$\begin{vmatrix} a_{11} & a_{12} & ma_{13} & a_{14} \\ a_{21} & a_{22} & ma_{23} & a_{24} \\ a_{31} & a_{32} & ma_{33} & a_{34} \\ a_{41} & a_{42} & ma_{43} & a_{44} \end{vmatrix} = m\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

In general, any factor of any row or column can be taken outside as a factor; or, vice versa, any factor outside can be put into the determinant as a multiplier of any row or column.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 3 \\ 8 & 5 & 6 \\ 14 & 8 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 7 & 8 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 4 & 6 \\ 3 & 6 & 12 \\ 5 & 15 & 30 \end{vmatrix} = 2 \cdot 3 \cdot 5 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 3 & 6 \end{vmatrix}$$

As a consequence of this, if two rows (columns) differ only by a constant factor, the value of the determinant is zero; for when the

factor is taken out, the resulting determinant has two rows (columns) identical.

(8) We can add together only determinants of the same order which differ only in one row (column). To do so add the corresponding elements in that row (column). Thus,

$$\begin{vmatrix} a_{11} & b_{12} & b_{13} \\ a_{21} & b_{22} & b_{23} \\ a_{31} & b_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} c_{11} & b_{12} & b_{13} \\ c_{21} & b_{22} & b_{23} \\ c_{31} & b_{32} & b_{33} \end{vmatrix} \quad \begin{vmatrix} a_{11}+c_{11} & b_{12} & b_{13} \\ a_{21}+c_{21} & b_{22} & b_{23} \\ a_{31}+c_{31} & b_{32} & b_{33} \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 3 & 5 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 2 & 3 & 0 \\ 2 & 1 & 2 & 1 \\ 3 & 5 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{vmatrix} \quad \begin{vmatrix} 3 & 3 & 4 & 1 \\ 2 & 1 & 2 & 1 \\ 3 & 5 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{vmatrix}$$

(9) It is possible without changing the value of the determinant to multiply the elements of any row (column) by the same constant and add the products to any other row (column). Thus,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} a_{11}+ma_{13} & a_{12} & a_{13} & a_{14} \\ a_{21}+ma_{23} & a_{22} & a_{23} & a_{24} \\ a_{31}+ma_{33} & a_{32} & a_{33} & a_{34} \\ a_{41}+ma_{43} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} ma_{13} & a_{12} & a_{13} & a_{14} \\ ma_{23} & a_{22} & a_{23} & a_{24} \\ ma_{33} & a_{32} & a_{33} & a_{34} \\ ma_{43} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \Delta$$

s the second determinant in the sum is zero.

(10) *The Laplace Expansion.* If $\mathbf{a} = | a_{ij} |$, then

$$\mathbf{a} = \sum_{j=1}^{n} a_{ij} A_{ij}$$

where the $a_{ij}$ must be taken from a single row (column), the $A_{ij}$ being the cofactors as previously defined. To illustrate this by a third-order determinant,

$$|a_{ij}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

can be expanded in six different ways:

$$\left.\begin{aligned}
&= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}
\end{aligned}\right\}\text{by rows}$$

$$\left.\begin{aligned}
&= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\
&= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\
&= a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}
\end{aligned}\right\}\text{by columns}$$

In general, this expansion can be written

$$\mathbf{a} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^{n} a_{ij} A_{ij}$$

where the $M_{ij}$ and $A_{ij}$ are as previously defined.

(11) Using the notation with cofactors, it is easy to show that

$$\begin{aligned}
\sum_{j=1}^{n} a_{ij} A_{kj} &= \mathbf{a} \qquad \text{if} \qquad i = k \\
&= 0 \qquad \text{if} \qquad i \neq k
\end{aligned}$$

The first is shown in (10). For the second, we note that in the preceding example

$$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = a_{11}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

(12) Determinants can be multiplied together. The process will first be indicated schematically. We shall define $I$ as the *unit determinant*, having units in the principal diagonal and zeros everywhere else. Also, define 0 as that determinant which has all elements zero.

$$I = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cdots = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$0 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = \text{etc.}$$

Then if $A$ and $B$ are determinants of the same order, $P$, their product, will be indicated in determinant form as

$$P = A \cdot B = A \cdot B - 0(-I) = \begin{vmatrix} A & -I \\ 0 & B \end{vmatrix} = \begin{vmatrix} A & 0 \\ -I & B \end{vmatrix}$$

This will be displayed in detail. With

$$A = |\, a_{ij} \,|, \qquad B = |\, b_{ij} \,|$$

then

$$P = A \cdot B = |\, a_{ij} \,| \cdot |\, b_{ij} \,| = \begin{vmatrix} |a_{ij}| & -I \\ 0 & |b_{ij}| \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & -1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & -1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & -1 \\ 0 & 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & 0 & \cdots & 0 & b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 & b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}$$

a determinant of the $2n$th order. From this, by the repeated use of the theorem of (9), we can develop the product into a determinant of the $n$th order. Multiply the $(n + 1)$st column

by $a_{11}$, the $(n+2)$d by $a_{21}$, etc., and add, respectively, to the first column, second column, etc. The result will be

$$P = A \cdot B = \begin{vmatrix} A & -I \\ 0 & B \end{vmatrix} \qquad \begin{vmatrix} 0 & -I \\ AB & B \end{vmatrix} = | AB |$$

where the elements of $| AB |$ are $\sum_{k=1}^{n} a_{ik}b_{kj}$ for $i, j = 1, 2 \quad n$.

Details are as follows, for a third-order set:

$$A \cdot B \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}$$

$$\equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} & -1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & -1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & -1 \\ 0 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 0 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{31} & b_{32} & b_{33} \end{vmatrix}$$

$$\begin{array}{cccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ & & & a_{11} & a_{21} & a_{31} \\ & & & a_{12} & a_{22} & a_{23} \\ & & & a_{13} & a_{23} & a_{33} \end{array}$$

$$= \begin{vmatrix} -a_{11} + a_{11} & -a_{12} + a_{12} & -a_{13} + a_{13} & -1 & 0 & 0 \\ -a_{21} + a_{21} & -a_{22} + a_{22} & -a_{23} + a_{23} & 0 & -1 & 0 \\ -a_{31} + a_{31} & -a_{32} + a_{32} & -a_{33} + a_{33} & 0 & 0 & -1 \\ a_{11}b_{11} + a_{21}b_{12} + a_{31}b_{13} & a_{12}b_{11} + a_{22}b_{12} + a_{32}b_{13} & a_{13}b_{11} + a_{23}b_{12} + a_{33}b_{13} & b_{11} & b_{12} & b_{13} \\ a_{11}b_{21} + a_{21}b_{22} + a_{31}b_{23} & a_{12}b_{21} + a_{22}b_{22} + a_{32}b_{23} & a_{13}b_{21} + a_{23}b_{22} + a_{33}b_{23} & b_{21} & b_{22} & b_{23} \\ a_{11}b_{31} + a_{21}b_{32} + a_{31}b_{33} & a_{12}b_{31} + a_{22}b_{32} + a_{32}b_{33} & a_{13}b_{31} + a_{23}b_{32} + a_{33}b_{33} & b_{31} & b_{32} & b_{33} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -I \\ \left| \sum_{j=1}^{n} a_{ji}b_{kj} \right| & B \end{vmatrix} = \left| \sum_{j=1}^{n} a_{ji}b_{kj} \right|$$

Evaluating this as if it were a third-order determinant, we have

$$P = \begin{matrix} a_{11}b_{11} + a_{21}b_{12} + a_{31}b_{13} \\ a_{11}b_{21} + a_{21}b_{22} + a_{31}b_{23} \\ a_{11}b_{31} + a_{21}b_{32} + a_{31}b_{33} \end{matrix} \quad \begin{matrix} a_{12}b_{11} + a_{22}b_{12} + a_{32}b_{13} \\ a_{12}b_{21} + a_{22}b_{22} + a_{32}b_{23} \\ a_{12}b_{31} + a_{22}b_{32} + a_{32}b_{33} \end{matrix} \quad \begin{matrix} a_{13}b_{11} + a_{23}b_{12} + a_{33}b_{13} \\ a_{13}b_{21} + a_{23}b_{22} + a_{33}b_{23} \\ a_{13}b_{31} + a_{23}b_{32} + a_{33}b_{33} \end{matrix}$$

The four forms are not all the same unless the factors have some sort of symmetry. Therefore, in general,

$$| AB | \neq | BA |$$

Somewhat different notations are used in various textbooks for products. One will meet

$$\begin{matrix} (abc)\widehat{\alpha} \\ \beta \\ \underset{\smile}{\gamma} \end{matrix} = \begin{vmatrix} a_\alpha & b_\alpha & c_\alpha \\ a_\beta & b_\beta & c_\beta \\ a_\gamma & b_\gamma & c_\gamma \end{vmatrix}$$

or

$$= \begin{vmatrix} (a \mid \alpha) & (a \mid \beta) & (a \mid \gamma) \\ (b \mid \alpha) & (b \mid \beta) & (b \mid \gamma) \\ (c \mid \alpha) & (c \mid \beta) & (c \mid \gamma) \end{vmatrix}, \text{ etc.}$$

(13) *Reciprocal; Adjoint.* These two are closely related, as follows: The *reciprocal* of $\mathbf{a} = | a_{ij} |$ is $\dfrac{1}{\mathbf{a}} = \dfrac{1}{| a_{ij} |}$, the *adjoint* of $\mathbf{a}$ is $A = | A_{ij} |$, where the $A_{ij}$ are the cofactors as previously defined. With this notation,

$$\frac{1}{\mathbf{a}} = \left| \frac{A_{ij}}{\mathbf{a}} \right|$$

or

$$\frac{1}{\mathbf{a}} = \frac{1}{\mathbf{a}^n} | A_{ij} | = \frac{1}{\mathbf{a}^n} \cdot A$$

whence

$$\mathbf{a}^n = \mathbf{a} \cdot A$$

or

$$\mathrm{a}^{n-1} = A$$

Here the adjoint is defined in terms of a. The proof of this relationship is shown by the use of the product theorem, as follows:

$$\begin{aligned} \mathrm{a} \cdot A &= | a_{ij} | \cdot | A_{ji} | \\ &= \left| \sum a_{ij} \cdot A_{ji} \right| \\ &= \mathrm{a}^n I \end{aligned}$$

for by (11) all elements other than diagonal ones are zero, and the diagonals all equal a.

(14) *Evaluation of Numerical Determinants.* The student should be able to evaluate numerical determinants quickly. The Laplace expansion in third and higher orders is space using and time consuming. The two following theorems are much preferable, when a high-order determinant is to be evaluated. The first, with

$$| a_{ij} | \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

then

$$| a_{ij} | = \frac{1}{(a_{11})^{n-2}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix}$$

Thus by the evaluation of $(n - 1)^2$ second-order determinants we are able to reduce an $n$th order determinant to a factor and a determinant of $(n - 1)$st order. This latter, in turn, by the

same theorem can be reduced to an $(n-2)$d, etc. An example:

$$\begin{vmatrix} 2 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & 3 & 1 & 4 \\ 0 & 1 & 0 & 1 \end{vmatrix} = \frac{1}{2^2}\begin{vmatrix} 2 & 1 & 2 \\ 2 & 0 & 4 \\ 2 & 0 & 2 \end{vmatrix}$$

$$= \frac{1}{2}\cdot\frac{1}{4}\begin{vmatrix} -2 & 4 \\ -2 & 0 \end{vmatrix} = \frac{1}{2}\cdot\frac{1}{4}\cdot 8 = 1$$

Another example:

$$\begin{vmatrix} 3 & 4 & 5 & 5 & 7 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 3 & 2 \\ 1 & 2 & 1 & 2 & 2 \end{vmatrix} = \frac{1}{3^3}\begin{vmatrix} 2 & 4 & 7 & 8 \\ -1 & -2 & -2 & -4 \\ 1 & -4 & -1 & -8 \\ 2 & -2 & 1 & -1 \end{vmatrix}, \text{ etc.}$$

This method can be used for any $a_{ij}$, but when the $a_{ij}$ is any other than the upper left-hand term, the matter of signs of the first-order determinants becomes confusing. It is an advantage if $a_{11} = 1$. If $a_{11} = 0$, it is of advantage to move columns or rows to obtain a term not zero in first place.

(15) The second rule:

$$|\,a_{ij}\,| = \frac{1}{\prod\limits_{j\neq 1,n} a_{ij}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{1,n-1} & a_{1n} \\ a_{2,n-1} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{1,n-1} & a_{1n} \\ a_{3,n-1} & a_{3n} \end{vmatrix} \\ \vdots & & & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{n2} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{1,n-1} & a_{1n} \\ a_{n,n-1} & a_{nn} \end{vmatrix} \end{vmatrix}$$

for any $i$ or $j$. This really needs no exemplification in the light of the preceding examples.

These two methods are much superior to any other method for evaluating numerical determinants and should be thoroughly learned by the student before proceeding to the solution of linear algebraic equations.

(16) *Rank of a Determinant.* If in a determinant all of its minors of order $r + 1$ are zero, while at least one of its minors of order $r$ is not zero, the determinant is said to have rank $r$.

(17) *Examples:*

(a) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{vmatrix}$ has rank 1 (b) $\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$ has rank 0

(c) $\begin{vmatrix} 3 & 2 & 4 & 6 \\ 1 & 3 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 6 & 4 & 8 & 12 \end{vmatrix}, r = 2$ (d) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, r = 4$

(e) $\begin{vmatrix} 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix}, r = 3$

(18) *Jacobians.* Given $F_j(x_i, y_j) = 0$

$$i, j = 1, 2, \cdots, n$$

Then the Jacobian is the determinant made up of terms as follows:

$$J \equiv \frac{\partial(x_i)}{\partial(y_i)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} & \frac{\partial x_3}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_3}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial y_n} & \frac{\partial x_2}{\partial y_n} & \frac{\partial x_3}{\partial y_n} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

(a) $$J \equiv (-1)^n \frac{\dfrac{\partial(F_i)}{\partial(x_i)}}{\dfrac{\partial(F_i)}{\partial(y_i)}}$$

(b) $$n = 1 \qquad -\frac{\partial F_i}{\partial x_i} = \frac{\partial F_i}{\partial y_i} \cdot \frac{\partial y_i}{\partial x_i}$$

(c) $$(-1)^n \frac{\partial(F_i)}{\partial(y_i)} = \frac{\partial(F_i)}{\partial(x_i)} \cdot \frac{\partial(x_i)}{\partial(y_i)}$$

(d) $$\frac{\partial(y_i)}{\partial(x_i)} \cdot \frac{\partial(x_i)}{\partial(y_i)} = 1$$

(e) If $J = 0$, the functions $F_j$ are not independent.

(19) *Hessians.* If in a Jacobian the $y_i$ are the $\partial f/\partial x_i$ of a function $f(x_i)$, then

$$H(f) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

is called the Hessian of $f(x_i)$.

(a) $H(f)$ is a symmetrical determinant, for

$$\frac{\partial^2 f}{\partial x_i \partial x_k} = \frac{\partial^2 f}{\partial x_k \partial x_i}$$

(b) If $\sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i} = 0$, then $H(f) = 0$.

(20) *Wronskians.* Let $f_i$ $(i = 1, 2, \cdots, n)$ be $n$ functions of a given variable. Suppose them connected by the linear relation

$$\sum_{i=1}^{n} a_i f_i = 0, \qquad a_i \text{ constants}$$

Differentiate successively

$$\sum_{i=1}^{n} a_i f_i^{(k)} = 0, \qquad k = 1, 2, \cdots, n-1$$

Eliminate the $a_i$ from the $n$ equations, obtaining the eliminant

$$W(f_i) = | f_j^{(i)} | = 0$$

This is called the Wronskian of the $f_i$.

**§17. Linear Algebraic Equations.**

(1) In general, a system of simultaneous linear algebraic equations will be written as follows:

$$\sum_{j=1}^{n} a_{ij} x_j = k_i \qquad i = 1, 2, \cdots, n \tag{a}$$

$a_{ij}$ real or complex constants

The determinant of the coefficients of the $x_j$ will be

$$A \equiv |\, a_{ij} \,|, \qquad \text{called the matrix determinant}$$

while $B \equiv \|\, A \mid k_i \,\|$ will be called the augmented matrix.

For a fourth-order system, the details will be

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= k_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= k_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= k_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= k_4 \end{aligned} \qquad (a)$$

Here

$$A \equiv \begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{matrix}$$

while

$$B \equiv \begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} & k_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & k_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & k_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & k_4 \end{matrix}$$

(2) *Cramer's Rule.* If, in ($a$), $A$ is not equal to zero, and there is one and only one set of solutions for the $x_j$, that set is given by

$$x_j = \frac{K_j}{A} \qquad j = 1, 2, \cdots, n$$

where the $K_j$ is $A$ with the $k_i$ in its $j$th column. This can be shown by direct substitution, as follows:

$$A \cdot x_j = x_j \left| \, a_{ij} \, \right| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j}x_j & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j}x_j & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3j}x_j & \cdots & a_{3n} \\ \cdot\cdot & \cdot\cdot & \cdots & \cdots & \cdots & \cdot\cdot \\ a_{n1} & a_{n2} & \cdots & a_{nj}x_j & \cdots & a_{nn} \end{vmatrix}$$

Then by the theorem of §15 (9), *i.e.*, by multiplying the other columns by the respective $x_k$ ($k = 1, 2, \cdots, n, \neq j$) and adding these products to the $j$th column, we have

$$A \cdot x_j = \begin{vmatrix} a_{11} & a_{12} & \cdots & \sum_{j=1}^{n} a_{1j}x_j & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \sum_{j=1}^{n} a_{2j}x_j & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & \sum_{j=1}^{n} a_{3j}x_j & \cdots & a_{3n} \\ \cdot\cdot & \cdot\cdot & \cdots & \cdots\cdots & \cdots & \cdot\cdot \\ a_{n1} & a_{n2} & \cdots & \sum_{j=1}^{n} a_{nj}x_j & \cdots & a_{nn} \end{vmatrix}$$

or, since $\sum_{j=1}^{n} a_{ij}x_j = k_i, \qquad i = 1, 2 \cdots n$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & k_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & k_2 & \cdots & a_{2n} \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot & \cdots & \cdot\cdot \\ a_{n1} & a_{n2} & \cdots & k_n & \cdots & a_{nn} \end{vmatrix} = K_j$$

For the fourth-order system, for instance, we should have, $j = 1$:

$$Ax_1 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} x_1 = \begin{vmatrix} a_{11}x_1 & a_{12} & a_{13} & a_{14} \\ a_{21}x_1 & a_{22} & a_{23} & a_{24} \\ a_{31}x_1 & a_{32} & a_{33} & a_{34} \\ a_{41}x_1 & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 & a_{12} & a_{13} & a_{14} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 & a_{22} & a_{23} & a_{24} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 & a_{32} & a_{33} & a_{34} \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$= \begin{vmatrix} k_1 & a_{12} & a_{13} & a_{14} \\ k_2 & a_{22} & a_{23} & a_{24} \\ k_3 & a_{32} & a_{33} & a_{34} \\ k_4 & a_{42} & a_{43} & a_{44} \end{vmatrix} = K_1$$

(3) The set of equations ($a$) may have no solution or may have an infinite number of solutions, depending on the relations

between the constants of the system. The conditions are best stated by Bôcher in his "Introduction to Higher Algebra," from which quotation is made.

(4) A necessary and sufficient condition for the system ($a$) to be consistent is that $A$ and $B$ have the same rank.

(5) If $A$ and $B$ have the same rank $r$ ($r < n$), then the values of $n - r$ of the unknowns may be assigned arbitrarily, and the others will then be uniquely determined. The $n - r$ unknowns may be chosen in any way provided that the matrix of the coefficients of the remaining unknowns is of rank $r$.

(6) *Examples:*

($a$)
$$\begin{aligned} y + z + u &= 1 \\ z + u + x &= 2 \\ u + x + y &= -1 \\ x + y + z &= -2 \end{aligned}$$

Here

$$A = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= -\begin{vmatrix} -2 & -1 \\ -1 & -2 \end{vmatrix} = -(4 - 1) = -3$$

$$K_1 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ -2 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} -2 & -1 & -1 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{vmatrix} = -\tfrac{1}{2}\begin{vmatrix} 0 & -2 \\ -3 & -1 \end{vmatrix}$$

$$= -\tfrac{1}{2}(-6) = 3 \qquad x = -1$$

Similarly,

$$K_2 = 6, \qquad K_3 = -3, \qquad K_4 = -6$$

whence

$$(x, y, z, w) = (-1, -2, 1, 2)$$

($b$)
$$\begin{aligned} x + 3y + 3z &= 9 \\ 2x + y + z &= 8 \\ 2x - y - z &= 4 \end{aligned}$$

$$A = \begin{vmatrix} 1 & 3 & 3 \\ 2 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix} = 0, \qquad \text{rank } 2$$

$$B = \begin{Vmatrix} 1 & 3 & 3 & 9 \\ 2 & 1 & 1 & 8 \\ 2 & -1 & -1 & 4 \end{Vmatrix}, \qquad \text{rank } 2$$

Solutions:

$$\begin{aligned} x + 3y &= 9 - 3z \\ 2x + \phantom{3}y &= 8 - \phantom{3}z \end{aligned}$$

$$A = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -5$$

$$Ax = \begin{vmatrix} 9 - 3z & 3 \\ 8 - z & 1 \end{vmatrix} = 9 - 3z - 24 + 3z = -15$$

$$Ay = \begin{vmatrix} 1 & 9 - 3z \\ 2 & 8 - z \end{vmatrix} = 8 - z - 18 + 6z = -10 + 5z$$

whence

$$x = 3, \qquad y = 2 - z$$

and

| | | | | |
|---|---|---|---|---|
| $z$ | 0 | 1 | 2 | . . . |
| $x$ | 3 | 3 | 3 | . . . |
| $y$ | 2 | 1 | 0 | . . . |

(c)

$$\begin{aligned} 2x - \phantom{2}y + 3z &= \phantom{-}1 \\ 4x - 2y - \phantom{3}z &= -3 \\ 2x - \phantom{2}y - 4z &= -4 \end{aligned}$$

(7) *Homogeneous Equations.* In the foregoing, if all the $k_i$ are zero, the procedure for solution is somewhat different. Write

$$\sum_{j=1}^{n} a_{ij}x_j = 0 \qquad i = 1, 2, \cdots, n \tag{b}$$

(8) In the first place, this set (b) always has one or more sets of solutions. It can have the solution $x_j = 0$, $j = 1, 2, \cdots, n$, called the trivial solution.

(9) For a solution other than the trivial one, however, a necessary and sufficient condition is that $A = 0$; *i.e.*, its rank is

less than $n$. When this is true, the theorem in (5) applies; and when arbitrary values are chosen for $n - r$ variables, Cramer's rule (2) will be used for obtaining the solutions.

(10) *Examples:*

(a)
$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 0 \\ 4x_1 - 12x_2 + 16x_3 &= 0 \\ 3x_1 - 9x_2 + 12x_3 &= 0 \end{aligned}$$
Rank of $A$ is 1, two unknowns arbitrary

(b)
$$\begin{aligned} x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 + 2x_3 &= 0 \\ x_1 + 5x_2 - x_3 &= 0 \end{aligned}$$
Rank of $A$ is 2, one unknown arbitrary

(c)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \\ 2x_1 + 5x_2 \phantom{+ x_3} &= 0 \end{aligned}$$
Rank of $A$ is 3, trivial solution

# CHAPTER V

## MATRICES

### §18. Definitions.

(1) *D-matrices* are rectangular arrays of terms that are polynomials in $D$ with real number coefficients. The double bars are used to set them off instead of single bars as in determinants. If $m$ is the number of rows, and $n$ the number of columns, $m$ is not necessarily equal to $n$, although when we use them in differential-equation theory, usually $m = n$. Examples of $D$-matrices are

$$(a)\quad \begin{Vmatrix} D^2 - kD + n^2 & k\omega \\ -k\omega & D^2 + kD + n^2 \end{Vmatrix}$$

$$(b)\quad \begin{Vmatrix} D & -r & q \\ r & D & -p \\ -q & p & D \end{Vmatrix}$$

$$(c)\quad \begin{Vmatrix} D^2 & \omega D & \dfrac{Ve}{m} \\ -\omega D & D^2 & 0 \end{Vmatrix}$$

(2) The *unit* matrix is defined as

$$I \equiv \begin{Vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{Vmatrix}$$

It may be of any order.

(3) If we have the matrix

$$M \equiv \| f_{ij} \| \equiv \begin{Vmatrix} f_{11} & f_{12} & f_{13} & \cdots & f_{1m} \\ f_{21} & f_{22} & f_{23} & \cdots & f_{2m} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ f_{m1} & f_{m2} & f_{m3} & \cdots & f_{mn} \end{Vmatrix}$$

then

$$kM \equiv \| kf_{ij} \| \equiv \begin{Vmatrix} kf_{11} & kf_{12} & \cdots & kf_{1m} \\ kf_{21} & kf_{22} & \cdots & kf_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ kf_{m1} & kf_{m2} & \cdots & kf_{mm} \end{Vmatrix}$$

(4) The *rank* of a $D$-matrix is the order of its largest determinant which is not identically zero. Examples:

(a) $\begin{Vmatrix} D^2 & \omega D & \dfrac{Ve}{m} \\ -\omega D & D^2 & 0 \end{Vmatrix}$, of rank 2

(b) $\begin{Vmatrix} 1 & D & -D^2 \\ -D & 1 & D \\ 1-D & D+1 & D-D^2 \end{Vmatrix}$, of rank 2

(c) $\begin{Vmatrix} D^2 & -2a\omega D & 0 \\ 2a\omega D & D^2 & 2b\omega D \\ 0 & -2b\omega D & D^2 \end{Vmatrix}$, of rank 3

(d) $\begin{Vmatrix} D^2 & D & 1 \\ D^3 & D^2 & D \end{Vmatrix}$, of rank 1

**§19. Transformations; an Example.**

(1) The main and perhaps only uses that we shall have for matrix transformations will be in showing how equivalent matrices are connected and to exhibit the invariant factors by the use of the normal form. To illustrate this, we shall use the matrix (*b*) under §18 (1), *viz.*,

$$\begin{Vmatrix} D & -r & q \\ r & D & -p \\ -q & p & D \end{Vmatrix}$$

This matrix is the set of algebraic and differential coefficients found in the set of differential equations of motion on moving axes generally, $p$, $q$, $r$ being angular velocities. [Taken from Lamb's "Higher Mechanics," p. 151 (3).]

$$\frac{dx}{dt} = ry - qz$$

$$\frac{dy}{dt} = pz - rx$$

$$\frac{dz}{dt} = qx - py$$

We set out to find the normal set of this system of equations. The procedure is as follows:

(2) Interchange first and third columns:

$$\begin{Vmatrix} D & -r & q \\ r & D & -p \\ -q & p & D \end{Vmatrix} \equiv \begin{Vmatrix} q & -r & D \\ -p & D & r \\ D & p & -q \end{Vmatrix}$$

(3) Now divide first row by $q$, and multiply second and third columns by $q$:

$$\equiv \begin{Vmatrix} 1 & -r & D \\ -p & qD & qr \\ D & pq & -q^2 \end{Vmatrix}$$

(4) Multiply first column by $r$, and add the products to the second column. Also, multiply the first column by $-D$, and add the products to the third column:

$$\equiv \begin{Vmatrix} 1 & 0 & 0 \\ -p & -pr + qD & pD + qr \\ D & rD + pq & -D^2 - q^2 \end{Vmatrix}$$

(5) Multiply first row by $p$ and add to the second row, and multiply first row by $-D$ and add to the third row:

$$\equiv \begin{Vmatrix} 1 & 0 & 0 \\ 0 & -pr + qD & pD + qr \\ 0 & rD + pq & -D^2 - q^2 \end{Vmatrix}$$

(6) We shall now transform the second-order matrix in the foregoing in similar manner. Multiply first column by $p$ and second column by $-q$, obtaining

$$\begin{matrix} -pr + qD & pD + qr \\ rD + pq & -D^2 - q^2 \end{matrix} \Bigg\Vert \equiv \Bigg\Vert \begin{matrix} -p^2r + pqD & -pqD - q^2r \\ prD + p^2q & qD^2 + q^3 \end{matrix}$$

(7) Add second column to first:

$$\left\| \begin{matrix} -p^2r - q^2r & -pqD - q^2r \\ qD + prD + p^2q + q^3 & qD^2 + q^3 \end{matrix} \right\|$$

(8) Divide first row by $-p^2r - q^2r$ and multiply second column by the same expression:

$$\equiv \left\| \begin{matrix} 1 & -pqD - q^2r \\ qD^2 + prD + p^2q + q^3 & -(qD^2 + q^3)(p^2r + q^2r) \end{matrix} \right\|$$

(9) Multiply first column by $pqD + q^2r$ and add to second column, and after that is done multiply first row by

$$-(qD^2 + prD + p^2q + q^3)$$

and add to the second row:

$$\equiv \left\| \begin{matrix} 1 & 0 \\ 0 & \begin{matrix} (qD^2 + prD + r^2q + q^3)(pqD + q^2r) \\ \qquad - (qD^2 + q^3)(p^2r + q^2r) \end{matrix} \end{matrix} \right\|$$

(10) Expand and collect terms in the last element:

$$\equiv \left\| \begin{matrix} 1 & 0 \\ 0 & pq^2[D^3 + (p^2 + q^2 + r^2)D] \end{matrix} \right\|$$

set $p^2 + q^2 + r^2 \equiv \alpha^2$, giving

$$\equiv \left\| \begin{matrix} 1 & 0 \\ 0 & pq^2(D^3 + \alpha^2 D) \end{matrix} \right\|$$

(11) Divide second column by $pq^2$, and factor:

$$\equiv \left\| \begin{matrix} 1 & 0 \\ 0 & D(D^2 + \alpha^2) \end{matrix} \right\|$$

(12) Replacing in the third-order matrix, we have

$$M \equiv \left\| \begin{matrix} D & -r & q \\ r & D & -p \\ -q & p & D \end{matrix} \right\| \equiv \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D(D^2 + \alpha^2) \end{matrix} \right\| \equiv N$$

(13) This latter matrix $N$ is called the *normal form* of $M$. It is also the matrix for the normal set of differential equations

which is said to be equivalent to the original set of differential equations. This normal matrix, therefore, plays an important role in the theory of the method of solution of the system of differential equations in exhibit. Our goal is nearly reached when we have exhibited this normal matrix, for it is the key to the procedure of solution which we give in a subsequent chapter. In the next section, however, we shall give the formal theory of permissible transformations on matrices, so that we may be able easily to solve systems of ordinary linear differential equations with constant coefficients.

**§20. The Formal Theory.**

(1) For a rigorous treatment of the following theory, the student is referred to the subject of "elementary divisors" or "invariant factors" in texts on higher algebra and determinants.*

(2) *Factors.* A factor of all first minors is a factor of the largest determinant of a $D$-matrix; for such determinant can be expanded by the Laplace method, as

$$\Delta \equiv \sum_{j=1}^{n}(-1)^{i+j}f_{ij}M_{ij}, \qquad \text{for any } i$$

Then if the $M_{ij}$ have a common factor $F$,

$$M_{ij} = F \cdot m_{ij}$$

whence

$$\frac{\Delta}{F} = \sum_{j=1}^{n}(-1)^{i+j}f_{ij}m_{ij}, \qquad \text{for any } i$$

(3) Similarly, if all second minors have a common factor, it is a factor of all first minors and hence of the determinant. The greatest common divisor of any set of $i$-rowed minors is thus a divisor of all $(i + 1)$-rowed minors. Indicating the G.C.D. by $G_i$,

$$\frac{G_{i+1}}{G_i} = E_{i+1}$$

the excess factors in the $(i + 1)$-rowed minors over those in the $i$-rowed ones.

* BÔCHER, M., "Introduction to Higher Algebra." MUIR, T., "The Theory of Determinants," vol. IV. SCOTT and MATTHEWS, "Theory of Determinants." MUTH, P., "Theorie u. Anwendung der Elementartheiler."

(4) It is thus evident that the $G_i$ are made up of the $E_i$; thus, with

$$
\begin{aligned}
G_0 &= 1 \\
G_1 &= E_1 \\
\frac{G_2}{G_1} &= E_2, \text{ etc.} \\
G_2 &= E_1 \cdot E_2 \\
G_3 &= E_1 \cdot E_2 \cdot E_3 \\
&\cdots\cdots\cdots \\
G_i &= E_1 \cdot E_2 \cdot E_3 \cdots E_i \\
&\cdots\cdots\cdots \\
\Delta &= \prod^n E_i = | f_{ij} |
\end{aligned}
$$

(5) *Elementary Transformations.* Three types of operations, called elementary transformations, may be performed upon a $D$-matrix without altering its essential nature, *viz.*,

*a.* The interchange of rows (columns).

*b.* The multiplication of each element of a row (column) by the same constant (not zero) or by a polynomial in $D$ (not identically zero).

*c.* The addition to the elements of a row (column) of the corresponding elements of any other row (column).

(6) *Equivalency.* Two $D$-matrices are equivalent if it is possible to pass from one to the other by elementary transformations. Thus, in the example of the last paragraph, all the matrices are equivalent.

(7) Elementary transformations do not affect the greatest common divisors of the set of all $i$-rowed determinants of a $D$-matrix; *i.e.*, such divisors are invariant under elementary transformations. Note that the determinants of the first and last matrices in the preceding example have $D^3 + (p^2 + q^2 + r^2)D$ as a factor, and unity is a factor of all second-order determinants in both, as well as all elements of both.

(8) From this it will be evident that the equivalency of two matrices means that their determinants contain the same factors for corresponding sets of minors. This is rigorously stated as follows:

(9) A necessary and sufficient condition for the equivalency of two $D$-matrices of the $n$th order is that

*a*. They have the same rank $r$, and

*b*. For every value of $i$, from 1 to $r$ inclusive, the $i$-rowed determinants of one matrix have the same greatest common divisors as the $i$-rowed determinants of the other. BÔCHER, *op. cit.*

(10) It is now obvious that the $G_i$ and $E_i$ are invariant under elementary transformations and that for equivalent matrices the $E_i$ are the *invariant factors*. These play an important part in the solution of systems of differential equations, as will be shown. The essential thing here is the procedure necessary to disclose quickly these $E_i$. The means are the elementary transformations, and the form that displays them is the *normal form*.

(11) *Normal Form.* We can now define the normal form and show how it is obtained. Given a $D$-matrix,

$$M \quad \begin{matrix} f_{11}(D) & f_{12}(D) & & f_{1n}(D) \\ f_{21}(D) & f_{22}(D) & & f_{2n}(D) \\ \cdot\ \cdot & \cdot\ \cdot & & \cdot\ \cdot \\ f_{n1}(D) & f_{n2}(D) & & f_{nn}(D) \end{matrix} \quad \equiv \| f_{ij}(D) \|, \text{ of rank } n$$

Its normal form will be an equivalent matrix having the $E_i$ of $M$ in its diagonal column and zeros elsewhere; thus,

$$M \equiv N \equiv \left\| \begin{matrix} E_1 & 0 & 0 & \cdot\ \cdot & \cdot & 0 \\ 0 & E_2 & 0 & & & 0 \\ 0 & 0 & E_3 & & & 0 \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot\ \cdot & E_{n-1} & 0 \\ 0 & 0 & 0 & \cdot\ \cdot & 0 & E_n \end{matrix} \right\|$$

(12) For a matrix of rank $r$ [$<n$] we would have

$$M \equiv N \equiv \begin{matrix} E_1 & 0 & & & & & \\ 0 & E_2 & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & E_r & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & & & & & \end{matrix}$$

since all $(r + 1)$-rowed determinants in $M$ are zero.

(13) *Disclosure of Invariant Factors.* The following procedures will most quickly transform any $D$-matrix into its normal form. Any one of the four may be used at any time in the process. Note that these are all elementary transformations.

*a.* Interchange of rows (columns), where any element $f_{ij}$ $[\neq 0, (i, j) \neq (1, 1)]$ is present of degree less than that of $f_{11}$. Purpose: to bring that $f_{ij}$ into first place.

*b.* Multiply any row (column) by any polynomial in $D$ and add the product to another row (column), when there is in the first row (column) an element $f_{1j}$, $f_{i1}$ in any other than first place of equal or greater degree than that of $f_{11}$. Purpose: to reduce the degree of that element $f_{1j}$ or $f_{i1}$ or to produce a zero in first row (column) in any other than first place.

*c.* Add any row $i$ (column $j$) to first row (column), when $f_{11}$ is not a factor of all other elements, and no $f_{ij}$ is of lower degree than that of $f_{11}$. Purpose: to reduce the degree of $f_{11}$.

*d.* Multiply any row (column) through by any constant $[\neq 0]$, at any time desired. Purpose: to simplify terms.

(14) The final result is to produce an equivalent matrix in which $f_{11}$ is of lowest degree possible and is a factor of all other elements, and the $f_{1j}$ and $f_{i1}$ are all zero. The $f_{11}$ left is then the $E_1$. The process is repeated for the matrix $\| f_{22}, f_{nn} \|$, etc., successively producing the $E_2$, $E_3$, etc.

(15) Going back now to our illustrative example of §19,

$$M \equiv \left\| \begin{matrix} D & -r & q & 1 & 0 & 0 \\ r & D & -p & 0 & 1 & 0 \\ -q & p & D & 0 & 0 & D(D^2 + \alpha^2) \end{matrix} \right\|$$

we may see that here we have

$$E_1 = 1, \quad E_2 = 1, \quad E_3 = D(D^2 + \alpha^2),$$
$$G_1 = 1, \quad G_2 = 1$$
$$G_3 = D(D^2 + \alpha^2) = \Delta = \text{determinant of the matrix}$$

(16) A second illustrative example may easily be verified by the student:

$$M \equiv \left\| \begin{matrix} D^2 & -2a\omega D & 0 \\ 2a\omega D & D^2 & 2b\omega D \\ 0 & -2b\omega D & D^2 \end{matrix} \right\|$$

$$\left.\begin{matrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D^2(D^2+\omega^2) \end{matrix}\right\| \equiv N, \qquad a^2+b^2=1$$

giving

$$\begin{aligned} E_1 &= D, & E_2 &= D, & E_3 &= D^2(D^2+\omega^2), \\ G_1 &= D, & G_2 &= D^2, & G_3 &= D^4(D^2+\omega^2) = \Delta = \quad M \end{aligned}$$

All first-order minors have a common factor $D$; all second-order minors have a common factor $D^2$; and the determinant

$$|\ M\ | = D^4(D^2+\omega^2).$$

(17) The student should transform the following matrices into the normal form and disclose the $E_i$ and $G_i$:

(c) $\left\| \begin{matrix} D+1 & D \\ D+3 & D+2 \end{matrix} \right\|$

(d) $\left\| \begin{matrix} D^2-3D+2 & D-1 \\ 1+D & D^2-5D+4 \end{matrix} \right.$

(e) $\left\| \begin{matrix} D^2 & nD \\ -nD & D^2 \end{matrix} \right\|$

(f) $\left\| \begin{matrix} D & -1 & -1 \\ -1 & D & -1 \\ -1 & -1 & D \end{matrix} \right\|$

(g) $\left\| \begin{matrix} D^2-1 & 1 & 1 \\ 1 & D^2-1 & 1 \\ 1 & 1 & D^2-1 \end{matrix} \right\|$

(h) $\left\| \begin{matrix} mD^2 & HeD \\ HeD & -mD^2 \end{matrix} \right\|$

(i) $\left\| \begin{matrix} D+\lambda_1 & 0 & 0 \\ \lambda_1 & -(D+\lambda_2) & 0 \\ 0 & \lambda_2 & -D \end{matrix} \right\|$

(j) $\left\| \begin{matrix} 7 & D-3 \\ 7D+63 & -36 \end{matrix} \right\|$

(k) $\left\| \begin{matrix} D-3 & 1 \\ -1 & D-1 \end{matrix} \right\|$

(l) $\left\| \begin{matrix} D-a_{11} & a_{12} \\ -a_{21} & D-a_{22} \end{matrix} \right\|.$

(m) $\begin{Vmatrix} D^2 - 4D & 1 - D \\ D + 6 & D^2 - D \end{Vmatrix}$

(n) $\begin{Vmatrix} 2D - 5 & 1 & 2 & 0 \\ 1 & 2D - 5 & 0 & 2 \\ 2 & 0 & 2D - 5 & 1 \\ 0 & 2 & 1 & 2D - 5 \end{Vmatrix}$

**§21. Symbolization and Classification.**

(1) Let us condense the normal matrix simply into its diagonal; *i.e.*,

$$N \equiv \begin{Vmatrix} E_1 & 0 & \cdots & 0 \\ 0 & E_2 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & E_n \end{Vmatrix} \equiv (E_1 \mid E_2 \mid \cdots \mid E_n)$$

If of rank $r < n$, we would have

$$N \equiv (E_1 \mid E_2 \mid \cdots \mid E_r \mid 0 \mid \cdots \mid 0)$$

(2) Now, since $E_i$ is a product of factors and $G_i = \Pi\, E_i$, the $G_i$ are the common factors of the $i$th-order minors; and since factors of any order minors are factors of all higher order minors, it follows that every factor in any $E_i$ is also contained in every $E_{i+p}$. Thus if we use a subletter to indicate degree, and $\Pi_k$ to indicate the factors in $E_i$, we may further symbolize the normal as

$$N \equiv (\Pi_r \mid \Pi_s \mid \cdots \mid \Pi_v \mid \Pi_w)$$

where $r + s + \cdots + v + w = m$, $m$ being the degree of the characteristic determinant $| M |$.

(3) When $E_1 \equiv \Pi_r$, we may factor it out of the matrix, thus:

$$N \equiv \Pi_r(1 \mid \Pi_s \mid \cdots \mid \Pi_{v-r} \mid \Pi_{w-r})$$

(4) We may say that $N$ is of class $[m \mid n]$, meaning of the $m$th degree and $n$th order. If of rank $r < n$, it would be $[m \mid n \mid r]$.

(5) We shall use the unit and zero matrices with a subletter for order, as $I_k$, $0_k$.

(6) Then we may disclose the following possibilities in the classification of matrices:

(a) $(1 \mid 1 \mid \ldots \mid 1 \mid \Pi_m)$

shortened into $(I_{m-1} \mid \Pi_m) \equiv (I_{m-1} \mid [m \mid 1])$

(b) $(1 \mid 1 \mid \cdots \mid 1 \mid \Pi_v \mid \Pi_{m-v}) \equiv (I_{m-2} \mid [m \mid 2])$

(c) $(1 \mid 1 \mid \cdots \mid 1 \mid \Pi_u \mid \Pi_{v-u} \mid \Pi_{m-v}) \equiv (I_{m-3} \mid [m \mid 3])$

$(n-1)$ $(1 \mid \Pi \quad \cdot \mid \Pi_{v-u} \mid \Pi_{m-v}\ ) \equiv (I_1 \mid [m \mid n-1])$

$(n)$ $(\Pi_r \mid \Pi \quad \Pi_{v-u} \mid \Pi_{m-v}\ )$

$\equiv \Pi_r(1 \mid \Pi_{s-r} \quad \Pi_{v-u-r} \mid \Pi_{m-v-r}) \equiv$

$\Pi_r(I_1 \mid [m-r \mid n-1])$

(7) *Examples:*

(a)
$$\begin{matrix} D & -r & q \\ r & D & -p \\ -q & p & D \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D(D^2+\alpha^2) \end{matrix}$$

$\equiv [3 \mid 3],$

$(1 \mid 1 \mid D(D^2+\alpha^2)) \equiv (I_2 \mid [3 \mid 1])$

(b)
$$\begin{matrix} D^2 & -2a\omega D & 0 \\ 2a\omega D & D^2 & 2b\omega D \\ 0 & -2b\omega D & D^2 \end{matrix} \qquad \begin{matrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D^2(D^2+4\omega^2) \end{matrix}$$

$\equiv [6 \mid 3], \qquad (D \mid D \mid D^2(D^2+4\omega^2))$

$\equiv D(1 \mid 1 \mid D(D^2+4\omega^2)) \equiv \Pi_1(I_2 \mid [3 \mid 1])$

(c)
$$\begin{matrix} D+1 & D \\ D+3 & D+2 \end{matrix} \qquad \begin{matrix} 1 & 0 \\ 0 & 2 \end{matrix}$$

$\equiv [0 \mid 2], \qquad (1 \mid 2) \equiv (I_1 \mid [0 \mid 1])$

(d)
$$\begin{matrix} D^2-3D+2 & D-1 \\ -\ \ D+1 & D^2-5D+4 \end{matrix}$$

$$\begin{matrix} D-1 & 0 \\ 0 & (D-1)(D-2)(D-4) \end{matrix}$$

$\equiv [4 \mid 2], \qquad (D-1 \mid (D-1)(D-2)(D-4))$

$\equiv (D-1)(1 \mid (D-2)(D-4)) \equiv \Pi_1(I_1 \mid [2 \mid 1])$

(e)
$$\left\| \begin{matrix} D^2 & nD \\ -nD & D^2 \end{matrix} \right\| \equiv \begin{matrix} D & 0 \\ 0 & D(D^2+n^2) \end{matrix}$$

$\equiv [4 \mid 2], \qquad (D \mid D^3+n^2D) \equiv D(1 \mid D^2+n^2)$

$\equiv \Pi_1(I_1 \mid [2 \mid 1])$

(f)
$$\left\| \begin{matrix} D & -1 & -1 \\ -1 & D & -1 \\ -1 & -1 & D \end{matrix} \right\| \equiv \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & D+1 & 0 \\ 0 & 0 & (D+1)(D-2) \end{matrix} \right\|$$

$\equiv [3 \mid 3], \qquad (1 \mid D+1 \mid (D+1)(D-2)) \equiv (I_1 \mid [3 \mid 2])$

(g)
$$\begin{Vmatrix} D^2-1 & 1 & 1 \\ 1 & D^2-1 & 1 \\ 1 & 1 & D^2-1 \end{Vmatrix} \equiv \begin{Vmatrix} 1 & 0 & 0 \\ 0 & D^2-2 & 0 \\ 0 & 0 & (D^2-2)(D^2+1) \end{Vmatrix}$$

$$\equiv [6 \mid 3], \qquad (1 \mid D^2-2 \mid (D^2-2)(D^2+1)) \equiv (I_1 \mid [6 \mid 2])$$

(8) The student should classify the rest of the matrices in §20 (17) by the symbolism above.

(9) All the foregoing are regular matrices; *i.e.*, the rank is $n$, the order of the matrices. When we come to consider singular matrices, *i.e.*, with rank $r$ less than $n$, we find first that the determinant of the matrix is zero, and there will also be $n-r$ zeros on the right end of the normal form. Thus,

(a)
$$\begin{Vmatrix} 1 & D & -D^2 \\ -D & 1 & D \\ 1-D & D+1 & D-D^2 \end{Vmatrix} \equiv \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$$

$$\equiv [0 \mid 3], \qquad (1 \mid 1 \mid 0) \equiv (I_2 \mid 0_1)$$

Here the rank of the matrix is 2, which says that there is at least one nonzero second-order determinant present. In this matrix, there are six second-order determinants not zero. Each of these can now be used as a nonsingular submatrix of second order, and that comes under the former classification.

(b)
$$\begin{Vmatrix} D^2 & -2\omega D & 0 & D^2 \\ 2\omega D & D^2 & 2\omega D & D \\ D^2+2\omega D & D^2-2\omega D & 2\omega D & D^2+D \\ D^2-2\omega D & -2\omega D - D^2 & -2\omega D & D^2-D \end{Vmatrix} \equiv \begin{Vmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}$$

$$\equiv [0 \mid 4], \qquad (D \mid D \mid 0 \mid 0) \equiv \Pi_1(I_2 \mid 0_2)$$

We shall return later to this classification of singular matrices under systems of linear differential equations.

CHAPTER VI

# SYSTEMS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

## §22. Physical Examples.

(1) Many physical situations are to be described by systems of ordinary linear differential equations with constant coefficients. We shall state a few of the well-known ones to show the student the necessity for knowing how to handle systems as well as single equations.

(2) *Electrical.* The transformer gives some interesting sets.

*a.* The single-phase transformer, with alternating current at one voltage fed into the primary circuit and a similar current at another voltage drawn from the secondary, would give the equations

$$L_1\frac{di_1}{dt} + M\frac{di_2}{dt} + R_1 i_1 \quad E_1$$

$$L_2\frac{di_2}{dt} + M\frac{di_1}{dt} + R_2 i_2 = 0$$

where $E_1 = E_0 \cos \omega t$, or $E_0 e^{j\omega t}$, $j = \sqrt{-1}$.

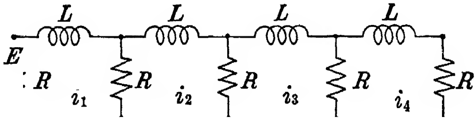
$L_1, L_2$ = inductances, in henrys.

$R_1, R_2$ = resistances, in ohms.

$M$ = mutual inductance.

$i_1, i_2$ = respective currents, in amperes.

*b.* A set of circuits




gives the set of equations

$$(Lp + 2R)i_1 - Ri_2 = E$$

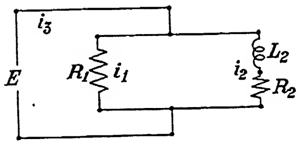
$$-Ri_1 + (Lp + 2R)i_2 - Ri_3 = 0$$

$$-Ri_2 + (Lp + 2R)i_3 - Ri_4 = 0$$
$$-Ri_3 + (Lp + 2R)i_4 = 0, \qquad p \equiv \frac{d}{dt}$$

whose matrix is

$$\left\| \begin{matrix} Lp + 2R & -R & 0 & 0 & E \\ -R & Lp + 2R & -R & 0 & 0 \\ 0 & -R & Lp + 2R & -R & 0 \\ 0 & 0 & -R & Lp + 2R & 0 \end{matrix} \right\|$$

*c.* The circuit




gives

$$i_3 = i_1 + i_2$$
$$R_1 i_1 = E$$
$$R_2 i_2 + L_2 \frac{di_2}{dt} = E$$

Here we may have $E = E_0$, or $E_0 e^{j\omega t}$

(3) *Miscellaneous.* *a.* A charged particle of charge $e$, mass $m$, in a field of uniform electrical intensity $X$ and magnetic intensity $H$ has the equations

$$m\frac{d^2x}{dt^2} = Xe - He\frac{dy}{dt}$$
$$m\frac{d^2y}{dt^2} = He\frac{dx}{dt}$$

whose matrix is

$$\left\| \begin{matrix} mp^2 & eHp & Xe \\ eHp & -mp^2 & 0 \end{matrix} \right\|$$

where $m$, $e$, $X$, $H$ are constants; with solutions, $\omega = \dfrac{He}{m}$

$$\omega Hx = X(1 - \cos \omega t)$$
$$\omega Hy = X(\omega t - \sin \omega t)$$

*b*. Free rotation of a body about a fixed axis with no external forces.

$$A\frac{dp}{dt} = -(C - A)\omega q$$

$$A\frac{dq}{dt} = (C - A)\omega p \qquad \text{[Lamb]}$$

*c*. Elliptic harmonic motion.

$$\frac{d^2x}{dt^2} = -mx, \qquad \frac{d^2y}{dt^2} = -my$$

*d*. Pendulum.

$$\text{Foucault:}\ \frac{d^2x}{dt^2} - 2\omega \sin \lambda\frac{dy}{dt} = -n^2x$$

$$\frac{d^2y}{dt^2} + 2\omega \cos \lambda\frac{dx}{dt} = -n^2y \qquad \text{[Lamb]}$$

$$\text{Blackburn:}\ \frac{d^2x}{dt^2} - 2\omega\frac{dy}{dt} - \omega^2x = -p^2x$$

$$\frac{d^2y}{dt^2} + 2\omega\frac{dx}{dt} - \omega^2y = -q^2y \qquad \text{[Lamb]}$$

*e*. Motion on moving axes generally has the matrix

$$\begin{Vmatrix} D & -r & q \\ r & D & -p \\ -q & p & D \end{Vmatrix} \qquad \text{[Lamb]}$$

*f*. Radioactive substances.

$$A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_n, \qquad n\text{th product } A_n \text{ stable.}$$

$$\frac{dA_0}{dt} = -\lambda_0A_0$$

$$\frac{dA_1}{dt} = \lambda_0A_0 - \lambda_1A_1$$

$$\frac{dA_2}{dt} = \lambda_1A_1 - \lambda_2A_2$$

$$\cdots\cdots$$

$$\frac{dA_n}{dt} = \lambda_{n-1}A_{n-1} - \lambda_nA_n$$

Here the matrix is

$$\begin{matrix} p+\lambda_0 & 0 & 0 & & \\ -\lambda_0 & p+\lambda_1 & 0 & & \\ 0 & -\lambda_1 & p+\lambda_2 & & \\ 0 & 0 & -\lambda_2 & & \\ & & & & \\ & & & p+\lambda_{n-1} & 0 \\ & & & -\lambda_{n-1} & p+\lambda_n \end{matrix}$$

*g*. Relative motion of a particle under gravity alone.

$$\begin{matrix} p^2 & -2\omega\ (\sin\lambda)p & 2\omega\ (\cos\lambda)p & 0 \\ 2\omega\ (\sin\lambda)p & p^2 & 0 & 0 \\ -2\omega\ (\cos\lambda)p & 0 & p^2 & -g \end{matrix} \qquad \text{[Lamb]}$$

Here $p \equiv \dfrac{d}{dt}$, $\omega$ = angular velocity, $\lambda$ = angle of initial direction, $g$ = gravity. For the unrestricted flight of a projectile we have the initial conditions

(i) $x = y = z = 0 = t$

(ii) $x' = u_0, \qquad y' = v_0, \qquad z' = \omega_0, \qquad t = 0$

(4) In the next three sections, we develop and illustrate the use of a simple all-comprehensive theory for the solution of any set of the type of differential equations above illustrated.

## §23. The Complementary Functions.

(1) Consider the system

$$\sum_{j=1}^{\cdot\cdot} F_{ij}(D)\cdot y_j = X_i \qquad i = 1, 2, \tag{A}$$

where the $F_{ij}(D)$ are of the form $\sum_k a_k D^k$, in which $D \equiv \dfrac{d}{dx}$; the $a_k$ are fixed constants in the complex field; and the $X_i$ are functions of the independent variable $x$.

(2) Dealing first with the homogeneous case, where all $X_i = 0$, we have the following Theorems I and II, governing its solution.

(3) **Theorem I: Each $\mathbf{y_j}$ and the general complementary function* satisfy the single differential equation**

$$|\ F_{ij}(D)\ |\cdot V = 0 \tag{B}$$

* This general complementary function is really the solution of the usual eliminant of the system [see (5) below].

**where the $| F_{ij}(D) |$ is the determinant of the system (A) and is not identically zero in D.**

(4) The proof of this theorem by the algebraic method is simple:

Assume the homogeneous case of $(A)$

$$\sum_{j=1}^{n} F_{ij}(D) \cdot y_j = 0 \qquad i = 1, 2, \cdots, n \qquad (A')$$

Multiply each of these equations by the cofactor $f_{ik}(D)$ of the $F_{ij}(D)$ in any $k$th column (Chap. IV, §16):

$$f_{ik}(D) \cdot \sum_{j=1}^{n} F_{ij}(D) \cdot y_j = 0 \qquad i, k, = 1, 2, \cdots, n$$

Adding for $i$, and regrouping,

$$\sum_{j=1}^{n} \left[ \sum_{i=1}^{n} f_{ik} \cdot F_{ij} \right] \cdot y_j = 0 \qquad k = 1, 2, \cdots, n$$

Of the coefficients of the $y_j$, all are zero except that in which $k = j$ [see IV §16 (11)]. We thus obtain

$$\left[ \sum_{i=1}^{n} f_{ij} \cdot F_{ij} \right] \cdot y_j = 0 \qquad j = 1, 2, \cdots \quad n$$

But

$$\sum_{i=1}^{n} f_{ij} F_{ij} \equiv | F_{ij} \qquad [§16 (10)]$$

so that

$$\sum_{i=1}^{n} f_{ij} \cdot F_{ij} \cdot y_j \equiv | F_{ij} | \cdot y_j = 0$$

and each $y_j$ is shown to be a solution for $(B)$.

(5) The solution $V$ of $(B)$ follows the method for the solution of a single equation (III §11). It will be called the *general complementary function* of the system $(A)$.

(6) The solution $V$ contains $m$ arbitrary constants and $m$ particular linearly independent solutions determined by the roots of the equation

$$F_{ij}(D) \mid = 0 \tag{C}$$

This equation demands that this determinant be not identically zero in $D$. The determinant is called the *characteristic determinant*, and Eq. ($C$) is called the *characteristic equation*, of the system ($A$).

(7) The degree of $\mid F_{ij}(D) \mid = 0$ is $m$, and this is the degree of the system ($A$). A rigorous proof of this statement has been given by Ince,* following Chrystal.

(8) **Theorem II. The $y_j$ are proportional to the cofactors of the operator coefficients of any *i*th row in the characteristic determinant, the constant of proportionality being the general complementary function V; *i.e.*,**

$$y_j = f_{ij} \cdot V, \qquad i = 1, 2, \cdot \cdot \cdot , n \tag{D}$$

(9) For proof we assume ($D$):

$$y_j = f_{ij} \cdot V$$

Multiply through by the $F_{kj}$ of any $k$th row of the matrix $\| F_{ij}(D) \|$, obtaining

$$F_{kj} \cdot y_j = F_{kj} \cdot f_{ij} \cdot V, \qquad j = 1, 2, \qquad , n, \text{ for any } i$$

Adding, for every $j$,

$$\sum_j F_{kj} \cdot y_j = \sum_j F_{kj} f_{ij} V \qquad i = 1, 2, \cdot \cdot \cdot , n, \text{ for any } k$$

Then, since

$$\sum F_{kj} \cdot f_{ij} \qquad F_{ij} \mid \quad \text{when} \quad k = i$$

$$= 0 \qquad \text{when} \quad k \neq i \text{ [IV §16 (11)]}$$

* Ince, E. L., "Ordinary Differential Equations," p. 144. Chrystal, George, *Trans. Roy. Soc. Edinburgh*, **38** (1898), 163.

we obtain a system of equations

$$\sum_j F_{ij} y_j = | \ F_{ij} \ | \cdot V^* \qquad i = 1, 2, \cdots, n \qquad (E)$$

The left-hand members of equations $(E)$ are the left-hand members of set $(A)$, so that

$$F_{ij}(D) \ | \cdot V = 0 \qquad (B)$$

Also, since they have just been shown to be equal to the left-hand member of $(B)$, it follows that $(E)$ are the solutions of $(B)$ and the complementary functions of $(A)$.

(10) *Fundamental Sets of Solutions.* When all the $y_j$ have been obtained, there will be present in them $m$ arbitrary constants and $m \cdot n$ particular integrals, the latter being either the $m$ integrals of $V$ or their derivatives. The original $m$ integrals in the general complementary function can be shown to be a fundamental set for the equation $(B)$ by evaluating their Wronskian. However, the integrals of the $y_j$ set must also be shown to be a fundamental set, for to be completely general they must be linearly independent of each other for each $y_j$. They must also be shown for $t = 0$ to have a matrix that has rank $n$. This will be illustrated in the examples. The proof of these statements is obvious from considerations of linear dependence.

**§ 24. The Arbitrary Constants.**

(1) The function $V$ contains the number of arbitrary constants demanded by the system $(A)$ as determined by the degree of the auxiliary equation $(C)$. When the cofactors $f_{ij}$ in $(D)$ operate on $V$, it will be found that some of the arbitrary constants may be lost to all $y_j$. This is always due to the presence of common factors in the $f_{ij}$, *i.e.*, $G_{n-1}$. Some means must be found, then, of restoring the lost arbitrary constants into the complementary functions, up to the necessary number.

(2) It is not necessary, as is usually done, to transform the system $(A)$ into an equivalent first-order system, but the normal form will be obtained to disclose the $G_i$. These factors and their positions in the normal matrix are a complete guide to the operations necessary to reduce the system $(A)$.

* SCOTT and MATTHEWS, "Theory of Determinants," p. 34.

(3) There are two cases which must be considered: (*a*) that where $| F_{ij} | = 0$ has all roots distinct and (*b*) that for multiple roots.

(4) *Case a: All Roots Distinct.* Here, the matrix is classified as

$$[m \mid n], \qquad (I_{n-1} \mid \Pi_m)$$

where $\prod_m \equiv \prod_{i=1}^{m} (D - \alpha_i)$. All first minors are distinct and have no common factors other than unity. The $f_{ij}$ are therefore distinct, with $G_{n-1} = 1$. Thus, not all the $f_{ij}$ in operating on $V$ can be made zero by the integrals due to the $\alpha_i$. We shall then have present after all operations all the necessary arbitrary constants in the $y_j$, which will then constitute the required set of complementary functions.

(5) *Example:* Consider Ex. (*a*) §21 (7), which is the matrix for the set of differential equations of motion on moving axes generally. The $p$, $q$, and $r$ are the angular velocities.*

$$\frac{dx}{dt} = ry - qz$$

$$\frac{dy}{dt} = pz - rx$$

$$\frac{dz}{dt} = qx - py$$

$$M \equiv \begin{matrix} D & -r & q \\ r & D & -p \\ -q & p & D \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D^3 + \alpha^2 D \end{matrix} \equiv N$$

$$\alpha^2 = p^2 + q^2 + r^2$$

System is of type

$$[3 \mid 3](1 \mid 1 \mid D^3 + \alpha^2 D) \equiv (I_2 \mid [3 \mid 1]$$

Here

$$\Delta \equiv | F_{ij}(D) | \equiv D^3 + \alpha^2 D$$

$$f_{11} = \begin{matrix} D & -p \\ p & D \end{matrix} = D^2 + p^2$$

* Lamb, "Higher Mechanics," p. 151 (3).

$$f_{12} = \begin{vmatrix} -p & r \\ D & -q \end{vmatrix} = pq - rD$$

$$f_{13} = \begin{vmatrix} r & D \\ -q & p \end{vmatrix} = rp + qD$$

$$f_{21} = \begin{vmatrix} p & D \\ -r & q \end{vmatrix} = pq + rD$$

$$f_{22} = \begin{vmatrix} D & -q \\ q & D \end{vmatrix} = D^2 + q^2$$

$$f_{23} = \begin{vmatrix} -q & p \\ D & -r \end{vmatrix} = rq - pQ$$

$$f_{31} = \begin{vmatrix} -r & q \\ D & -p \end{vmatrix} = rp - qD$$

$$f_{32} = \begin{vmatrix} q & D \\ -p & r \end{vmatrix} = rq + pD$$

$$f_{33} = \begin{vmatrix} D & -r \\ r & D \end{vmatrix} = D^2 + r^2$$

Then

$$| F_{ij} | \cdot V = 0 \qquad \text{is} \qquad (D^3 + \alpha^2 D)V = 0$$

and

$$V = C_{i1} + C_{i2} \cos \alpha t + C_{i3} \sin \alpha t$$

Now, if $x = x_{i1}$, $y = x_{i2}$, $z = x_{i3}$, then

$$\frac{x_{i1}}{f_{i1}} = \frac{x_{i2}}{f_{i2}} = \frac{x_{i3}}{f_{i3}} = V_i$$

Now, for $i = 1$

$$\frac{x_{11}}{D^2 + p^2} = \frac{x_{12}}{pq - rD} = \frac{x_{13}}{rp + qD} = C_{11} + C_{12} \cos \alpha t + C_{13} \sin \alpha t$$

$$x_{11} = (D^2 + p^2)V_1 = p^2 C_{11} + C_{12}(p^2 - \alpha^2) \cos \alpha t + C_{13}(p^2 - \alpha^2) \sin \alpha t$$

$$x_{12} = (pq - rD)V_1 = pqC_{11} + C_{12}(pq \cos \alpha t + r\alpha \sin \alpha t) + C_{13}(pq \sin \alpha t - r\alpha \cos \alpha t)$$

$$x_{13} = (rp + qD)V_1 = rpC_{11} + C_{12}(rp \cos \alpha t - q\alpha \sin \alpha t) + C_{13}(rp \sin \alpha t + q\alpha \cos \alpha t)$$

For $t = 0$,

$$
\begin{aligned}
x_{11} &= p^2C_{11} + (p^2 - \alpha^2)C_{12} \\
x_{12} &= pqC_{11} + pqC_{12} - r\alpha C_{13} \\
x_{13} &= rpC_{11} + rpC_{12} + q\alpha C_{13}
\end{aligned}
$$

The determinant of this will be

$$
\begin{vmatrix} p^2 & p^2 - \alpha^2 & 0 \\ pq & pq & -r\alpha \\ rp & rp & q\alpha \end{vmatrix} \equiv p\alpha^3(q^2 + r^2) \neq 0
$$

For $i = 2$,

$$
\frac{x_{21}}{pq + rD} = \frac{x_{22}}{D^2 + q^2} = \frac{x_{23}}{rq - pD} = V_2
$$

$$
\begin{aligned}
x_{21} &= (pq + rD)V_2 = pqC_{21} + C_{22}(pq \cos \alpha t - r\alpha \sin \alpha t) \\
&\qquad + C_{23}(pq \sin \alpha t + r\alpha \cos \alpha t) \\
x_{22} &= (D^2 + q^2)V_2 = q^2C_{21} + C_{22}(q^2 - \alpha^2) \cos \alpha t \\
&\qquad + C_{23}(q^2 - \alpha^2) \sin \alpha t \\
x_{23} &= (rq - pD)V_2 = rqC_{21} + C_{22}(rq \cos \alpha t + p\alpha \sin \alpha t) \\
&\qquad + C_{23}(rq \sin \alpha t - p\alpha \cos \alpha t)
\end{aligned}
$$

For $t = 0$,

$$
\begin{aligned}
x_{21} &= pqC_{21} + pqC_{22} + r\alpha C_{23} \\
x_{22} &= q^2C_{21} + (q^2 - \alpha^2)C_{22} \\
x_{23} &= rqC_{21} + rqC_{22} - p\alpha C_{23}
\end{aligned}
$$

The determinant of this will be

$$
\begin{vmatrix} pq & pq & r\alpha \\ q^2 & q^2 - \alpha^2 & 0 \\ rq & rq & -p\alpha \end{vmatrix} \equiv q\alpha^3(p^2 + r^2) \neq 0
$$

For $i = 3$,

$$
\frac{x_{31}}{rp - qD} = \frac{x_{32}}{rq + pD} = \frac{x_{33}}{D^2 + r^2} = V_3
$$

$$
\begin{aligned}
x_{31} &= (rp - qD)V_3 = rpC_{31} + C_{32}(rp \cos \alpha t + q\alpha \sin \alpha t) \\
&\qquad + C_{33}(rp \sin \alpha t - q\alpha \cos \alpha t) \\
x_{32} &= (rq + pD)V_3 = rqC_{31} + C_{32}(rq \cos \alpha t - p\alpha \sin \alpha t) \\
&\qquad + C_{33}(rq \sin \alpha t + p\alpha \cos \alpha t)
\end{aligned}
$$

$$x_{33} = (D^2 + r^2)V_3 = r^2C_{31} + C_{32}(r^2 - \alpha^2)\cos\alpha t + C_{33}(r^2 - \alpha^2)\sin\alpha t$$

For $t = 0$,

$$\begin{aligned} x_{31} &= rpC_{31} + rpC_{32} - q\alpha C_{33} \\ x_{32} &= rqC_{31} + rqC_{32} + p\alpha C_{33} \\ x_{33} &= r^2C_{31} + (r^2 - \alpha^2)C_{32} \end{aligned}$$

The determinant will be

$$\begin{vmatrix} rp & rp & -q\alpha \\ rq & rq & p\alpha \\ r^2 & r^2 - \alpha^2 & 0 \end{vmatrix} \equiv r\alpha^3(p^2 + q^2) \neq 0$$

We now have three fundamental sets of solutions which we summarize below. All others can be written in terms of these.

*Summary:*

$$\text{I.}\quad \begin{cases} x_{11} = C_{11}p^2 + C_{12}(p^2 - \alpha^2)\cos\alpha t + C_{13}(p^2 - \alpha^2)\sin\alpha t \\ x_{12} = C_{11}pq + C_{12}(pq\cos\alpha t + r\alpha\sin\alpha t) \\ \qquad\qquad + C_{13}(pq\sin\alpha t - r\alpha\cos\alpha t) \\ x_{13} = C_{11}rp + C_{12}(rp\cos\alpha t - q\alpha\sin\alpha t) \\ \qquad\qquad + C_{13}(rp\sin\alpha t + q\alpha\cos\alpha t) \end{cases}$$

$$\text{II.}\quad \begin{cases} x_{21} = C_{21}pq + C_{22}(pq\cos\alpha t - r\alpha\sin\alpha t) \\ \qquad\qquad + C_{23}(pq\sin\alpha t + r\alpha\cos\alpha t) \\ x_{22} = C_{21}q^2 + C_{22}(q^2 - \alpha^2)\cos\alpha t + C_{23}(q^2 - \alpha^2)\sin\alpha t \\ x_{23} = C_{21}rq + C_{22}(rq\cos\alpha t + p\alpha\sin\alpha t) \\ \qquad\qquad + C_{23}(rq\sin\alpha t - p\alpha\cos\alpha t) \end{cases}$$

$$\text{III.}\quad \begin{cases} x_{31} = C_{31}rp + C_{32}(rp\cos\alpha t + q\alpha\sin\alpha t) \\ \qquad\qquad + C_{33}(rp\sin\alpha t - q\alpha\cos\alpha t) \\ x_{32} = C_{31}rq + C_{32}(rq\cos\alpha t - p\alpha\sin\alpha t) \\ \qquad\qquad + C_{33}(rq\sin\alpha t + p\alpha\cos\alpha t) \\ x_{33} = C_{31}r^2 + C_{32}(r^2 - \alpha^2)\cos\alpha t + C_{33}(r^2 - \alpha^2)\sin\alpha t \end{cases}$$

(6) Under Case *a* comes also by a suitable initial substitution that type under Case *b* whose matrix contains a factor in all elements. For instance,

$$(D - 1 \mid (D - 1)(D - 2)(D - 4)) \equiv (D - 1)(1 \mid (D - 2)(D - 4))$$

Here a substitution of $(D - 1)y_j = z_j$ is suggested at the start. Then the system $(1 \mid (D-2)(D-4))$ is regular in $z_j$, after the solution of which, we have

$$y_j = \frac{1}{D-1}z_j + \frac{1}{D-1}\cdot 0$$

Similar examples are

(e) $$D(1 \mid D^2 + n^2)$$

(f) $$D(1 \mid 1 \mid D^3 + 4\omega^2 D)$$

(7) Working (e). This system completely stated is

$$\frac{d^2y_1}{dt^2} + n\frac{dy_2}{dt} = 0$$

$$-n\frac{dy_1}{dt} + \frac{d^2y_2}{dt^2} = 0 \qquad \text{[Ince, 157 (4iv) XVII, 49]}$$

$$\begin{Vmatrix} D^2 & nD \\ -nD & D^2 \end{Vmatrix} \equiv \begin{Vmatrix} D & 0 \\ 0 & D(D^2+n^2) \end{Vmatrix}$$

$$\equiv [4 \mid 2](D \mid D(D^2+n^2)) \equiv D(1 \mid D^2+n^2)$$

Substitute

$$Dy_1 = z_1, \qquad Dy_2 = z_2$$

The set becomes

$$\frac{dz_1}{dt} + nz_2 = 0$$

$$-nz_1 + \frac{dz_2}{dt} = 0$$

the solution of which comes under Case *a*:

$$\begin{matrix} D & n \\ -n & D \end{matrix} \qquad \begin{matrix} 1 & 0 \\ 0 & D^2+n^2 \end{matrix}$$

$$\equiv [2 \mid 2](1 \mid D^2+n^2)$$

when $z_1$ and $z_2$ are found, then

$$y_1 = \frac{1}{D}(z_1 + 0)$$

$$y_2 = \frac{1}{D}(z_2 + 0)$$

(8) *Case b: Multiple Roots.* Every root $D = \alpha_k$ of multiplicity $s + 1$ in $| F_{ij}(D) | = 0$ gives this determinant a factor

$$(D - \alpha_k)^{s+1}.$$

Then $D - \alpha_k$ may or may not appear in all the first minors, or cofactors, $f_{ij}$. When it does not appear in all $f_{ij}$, we have Case $a$, perfectly regular. When it does appear, it can have the degree $s$ at most.* Let $s$, however, represent the degree to which it does appear. Here the matrix is classified as

$$[m \mid n](1 \mid 1 \mid \cdots \mid 1 \mid \Pi_s \mid \Pi_m) \equiv (I_{n-2} \mid \Pi_s[m - s \mid 2])$$

(9) We shall now use Eqs. $(D)$,

$$\frac{y_j}{f_{ij}} = V \qquad \begin{array}{l} j = 1, 2, \cdots, n \\ \text{for any } i \end{array}$$

and first disclose the factors $(D - \alpha_k)^s$, *viz.*,

$$\frac{y_j}{(D - \alpha_k)^s \beta_{ij}} = V \qquad \begin{array}{l} j = 1, 2, \cdots, n \\ \text{for any } i \end{array}$$

Now, operate on the left with $(D - \alpha_k)^s$, obtaining

$$\frac{(D - \alpha_k)^s y_j}{(D - \alpha_k)^s \beta_{ij}} = (D - \alpha_k)^s \cdot V$$

Using $(D - \alpha_k)^s \cdot (D - \alpha_k)^{-s} \equiv 1$, we have

$$\frac{y_j}{\beta_{ij}} = (D \quad \alpha_k)^s \cdot V \tag{F}$$

Operate now by $(D - \alpha_k)^p$ and use

$$(D - \alpha_k)^p \cdot y_j = z_j \qquad \begin{array}{l} 1 \leqq p \leqq s \\ p \text{ an integer} \end{array}$$

obtaining

$$\frac{z_j}{\beta_{ij}} = (D \quad \alpha_k)^{s+p} \cdot V = W \tag{G}$$

Then

$$z_j = \beta_{ij} \cdot W \qquad \begin{array}{l} j = 1, 2, \cdots, n \\ \text{for any } i \end{array}$$

* DICKSON, "First Course in the Theory of Equations," p. 61.

The function $W$ will contain $s + p$ less arbitrary constants than $V$. The $\beta_{ij}$ will not have a common factor, and when they operate on $W$ they cannot all be made zero by the multiple root. As a matter of fact, $W$ will often not contain any such integral. The set of $z_j$ will thus contain $n - (s + p)$ arbitrary constants.

(10) Now, substitute the $z_j$ back in

$$y_j = (D - \alpha_k)^{-p} z_j \tag{H}$$

obtaining

$$y_j = (D - \alpha_k)^{-p}[\beta_{ij} W + C] \qquad j = 1, 2, \tag{K}$$

It is necessary to use the inverse operation on zero only $q$ times before obtaining the $n$ arbitrary constants desired, where $q$ is an integer satisfying the equation

$$s + p = p \cdot q$$

This produces $q$ of the $y_j$.

(11) The rest of the $y_j$, $n - q$ in number, are to be obtained in terms of the $q$ already found, as linear combinations of them, from the proportions

$$\frac{y_h}{\beta_{ih}} \qquad \begin{matrix} j, h, = 1, 2, \cdot \cdot \\ j \neq h \end{matrix}$$

thus:

$$\frac{y_j}{\beta_{ij}} \qquad \frac{\Sigma y_h}{\Sigma \beta_{ih}} \qquad \begin{matrix} h = 1, 2, \cdot \cdot \cdot , q \\ j = q + 1, q + 2, \qquad\qquad n \\ \text{for any } i \end{matrix}$$

so that we have

$$y_j = \frac{\beta_{ij}}{\Sigma \beta_{ih}} \Sigma y_h \qquad \begin{matrix} h = 1, 2, \cdot \cdot \cdot , q \\ j = q + 1, q + 2, \qquad\qquad , n \end{matrix} \tag{L}$$

Here we do not use a zero because we need no more arbitrary constants. The results $(K)$ and $(L)$ will be the complete solution, having only the proper number of arbitrary constants.

(12) The equation $s + p = p \cdot q$ fails when $p = 0$, in which case it should become

$$s = s \cdot q$$

whence $q = 1$, which states that but one integration is necessary to replace the lost arbitrary constants.

(13) *Example:*

$$\begin{matrix} D^2 & nD & D & 0 \\ -nD & D^2 & 0 & D(D^2 + n^2) \end{matrix} \qquad D(1 \mid [2 \mid 2])\ [4 \mid 2];$$

$$F_{ij}(D) \mid \cdot V = D^2(D^2 + n^2) \cdot V = 0, \qquad V = \frac{1}{D^2(D^2 + n^2)} \cdot 0$$

$$f_{11} = D^2, \qquad f_{12} = nD$$
$$f_{21} = -nD, \qquad f_{22} = D^2$$
$$\frac{y_{i1}}{f_{i1}} = \frac{y_{i2}}{f_{i2}} = V_i$$

First set:

$$\frac{y_{11}}{D^2} = \frac{y_{12}}{nD} = V_1 = \frac{1}{D^2(D^2 + n^2)} \cdot 0$$
$$= C_{11} + C_{12}t + C_{13} \cos nt + C_{14} \sin nt$$
$$y_{11} = D^2V_1 = -n^2(C_{13} \cos nt + C_{14} \sin nt)$$
$$y_{12} = nDV_1 = nC_{12} - n(C_{13} \sin nt - C_{14} \cos nt)$$

which contain only three arbitrary constants, an insufficient number. We shall, therefore, apply the operator $D$, $s = 1$, and $D^{-1}D \equiv 1$.

$$\frac{y_{11}}{D} \qquad n \quad - \frac{1}{D(D^2 + n^2)} \cdot 0$$

Again, since $s + p = p \cdot q$, $s = 1$ gives $1 + p = p \cdot q$; we have $1 + 1 = 1 \cdot 2$, and $q = 2$. Then

$$\frac{z_{11}}{D} = \frac{z_{12}}{n} = \frac{1}{D^2 + n^2} \cdot 0 = W_1$$
$$= C_{13} \cos nt + C_{11} \sin nt$$
$$z_{11} = D \cdot W_1 = -n(C_{13} \sin nt - C_{14} \cos nt)$$
$$z_{12} = nW_1 = n(C_{13} \cos nt + C_{14} \sin nt)$$

Then

$$y_{11} = \frac{1}{D}[z_{11} + 0] = C_{13} \cos nt + C_{14} \sin nt + C_{11}$$

$$y_{12} = \frac{1}{D}[z_{12} + 0] = C_{13} \sin nt - C_{14} \cos nt + C_{12}$$

which now have four arbitrary constants, the proper number.

For $t = 0$,

$$\begin{aligned} y_{11} &= C_{13} + C_{11} \\ y_{12} &= -C_{14} + C_{12} \end{aligned} \qquad \left\| \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{matrix} \right\|, \qquad \text{rank } 2$$

This is a fundamental set. Its Wronskian is $-n^2 \neq 0$.

Second set:

$$\frac{y_{21}}{-nD} = \frac{y_{22}}{D^2} = \frac{1}{D^2(D^2 + n^2)} \cdot 0$$
$$= C_{21} + C_{22}t + C_{23} \cos nt + C_{24} \sin nt$$
$$\frac{y_{21}}{-n} = \frac{y_{22}}{D} = \frac{1}{D(D^2 + n^2)} \cdot 0$$
$$\frac{z_{21}}{-n} = \frac{z_{22}}{D} = \frac{1}{D^2 + n^2} \cdot 0 = M_1 = C_{23} \cos nt + C_{24} \sin nt$$
$$\begin{aligned} z_{21} &= -nC_{23} \cos nt - nC_{24} \sin nt \\ z_{22} &= -nC_{23} \sin nt + nC_{24} \cos nt \end{aligned}$$

Then

$$y_{21} = \frac{1}{D}[z_{21} + 0] = -C_{23} \sin nt + C_{24} \cos nt + C_{21}$$

$$y_{22} = \frac{1}{D}[z_{22} + 0] = C_{23} \cos nt + C_{24} \sin nt + C_{22}$$

For $t = 0$,

$$\begin{aligned} y_{21} &= C_{24} + C_{21} \\ y_{22} &= C_{23} + C_{22} \end{aligned} \qquad \left\| \begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{matrix} \right\|, \qquad \text{rank } 2$$

This is the second fundamental set.

Each satisfies completely the original set of differential equations and is completely general, with four arbitrary constants.

*Summary:*

| | $y_{i1}$ | $y_{i2}$ |
|---|---|---|
| I. | $C_{11} + C_{13}\cos nt + C_{14}\sin nt$ | $C_{12} + C_{13}\sin nt - C_{14}\cos nt$ |
| II. | $C_{21} - C_{23}\sin nt + C_{24}\cos nt$ | $C_{22} + C_{23}\cos nt + C_{24}\sin nt$ |

*Another Example:*

$$\left\|\begin{array}{ccc} D^2 & -2a\omega D & 0 \\ 2a\omega D & D^2 & 2b\omega D \\ 0 & -2b\omega D & D^2 \end{array}\right\| \equiv \left\|\begin{array}{ccc} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D^2(D^2+4\omega^2) \end{array}\right\|$$

$\equiv [6 \mid 3](D \mid D \mid D^2(D^2+n^2)) \equiv D(I_2 \mid [3 \mid 1]),\ a^2+b^2=1$

$f_{11} = D^2(D^2+4b^2\omega^2),\qquad f_{21} = 2a\omega D^3,\qquad f_{31} = -4ab\omega^2 D^2$

$f_{12} = -2a\omega D^3,\qquad f_{22} = D^4,\qquad f_{32} = -2b\omega D^3$

$f_{13} = -4ab\omega^2 D^2,\qquad f_{23} = 2b\omega D^3,\qquad f_{33} = D^2(D^2+4a^2\omega^2)$

$$|F_{ij}(D)| \cdot V = D^4(D^2+4\omega^2)\cdot V = 0, \qquad V = \frac{1}{D^4(D^2+4\omega^2)}\cdot 0$$

Here

$$m = 6, \qquad n = 3, \qquad G_2 = D^2, \qquad s = 2$$

Then $s + p = pq$ gives two possibilities for the arbitrary constants, *viz.*,

$$p = 1, \qquad q = 3, \qquad \text{or} \qquad p = 2, \qquad q = 2$$

We shall use the second possibility, as it offers the most interesting and instructive method.

$$\frac{y_{i1}}{f_{i1}} = \frac{y_{i2}}{f_{i2}} = \frac{y_{i3}}{f_{i3}} = V_i$$

Using $i = 1$,

$$\frac{y_{11}}{D^2(D^2+4b^2\omega^2)} = \frac{y_{12}}{-2a\omega D^3} = \frac{y_{13}}{-4ab\omega^2 D^2} = \frac{1}{D^4(D^2+4\omega^2)}\cdot 0$$

Since $s = 2, G_2 = D^2$, operate through by $D^2$, and use $D^{-2}D^2 \equiv 1$.

$$\frac{y_{11}}{D^2+4b^2\omega^2} = \frac{y_{12}}{-2a\omega D} = \frac{y_{13}}{-4ab\omega^2} = \frac{1}{D^2(D^2+4\omega^2)}\cdot 0$$

Now, $p = 2$, so operate again, and substitute $z_{1j} = D^2 y_{1j}$.

$$\frac{z_{11}}{D^2+4b^2\omega^2} = \frac{z_{12}}{-2a\omega D} = \frac{z_{13}}{-4ab\omega^2} = \frac{1}{D^2+4\omega^2}\cdot 0 = W_1$$

Then

$$\begin{aligned} z_{11} &= (D^2 + 4b^2\omega^2)W_1 \\ &= -4a^2\omega^2(C_{15} \cos 2\omega t + C_{16} \sin 2\omega t) \\ z_{12} &= (-2a\omega D)W_1 \\ &= -4a\omega^2(-C_{15} \sin 2\omega t + C_{16} \cos 2\omega t) \\ z_{13} &= -4ab\omega^2(C_{15} \cos 2\omega t + C_{16} \sin 2\omega t) \end{aligned}$$

Now, since $q = 2$, we integrate two of the foregoing (any two, say $z_{11}$, $z_{12}$) by the inverse

$$z_{1j} = D^2 y_{1j}$$

$$\begin{aligned} y_{11} &= \frac{1}{D^2}[z_{11} + 0] \\ &= a^2(C_{15} \cos 2\omega t + C_{16} \sin 2\omega t) + C_{11} + C_{12}t \\ y_{12} &= \frac{1}{D^2}[z_{12} + 0] \\ &= a(-C_{15} \sin 2\omega t + C_{16} \cos 2\omega t) + C_{13} + C_{14}t \end{aligned}$$

Now obtain $y_{13}$ in terms of $y_{11}$ and $y_{12}$:

$$\frac{y_{13}}{-4ab\omega^2} = \frac{y_{11} + y_{12}}{D^2 + 4b^2\omega^2 - 2a\omega D}$$

$$y_{13} = \frac{-4ab\omega^2}{D^2 - 2a\omega D + 4b^2\omega^2}(y_{11} + y_{12})$$

Omitting the actual operations, the result is

$$\begin{aligned} y_{13} = (C_{11} + C_{13})\frac{a}{b\omega} - (C_{12} + C_{14})\left(\frac{a}{b}t + \frac{a}{2b^3\omega}\right) \\ - (C_{15}a^2 + C_{16}a)\frac{1}{2\omega(1 + a^2)}(-2\omega \sin 2\omega t + 2a\omega \cos 2\omega t) \\ - (C_{16}a^2 - C_{15}a)\frac{1}{2\omega(1 + a^2)}(2\omega \cos 2\omega t + 2a\omega \sin 2\omega t) \end{aligned}$$

This is one fundamental set. There are two others, which we shall not obtain. This is sufficient to illustrate the method completely. Note that there are six arbitrary constants as required.

(14) Examples for the student:

(*d*)

$$\begin{matrix} D^2 - 3D + 2 & D - 1 \\ -D + 1 & D^2 - 5D + 4 \end{matrix}$$

(f) $$\left\| \begin{matrix} D & -1 & -1 \\ -1 & D & -1 \\ -1 & -1 & D \end{matrix} \right\|$$

(g) $$\left\| \begin{matrix} D^2 - 1 & 1 & 1 \\ 1 & D^2 - 1 & 1 \\ 1 & 1 & D^2 - 1 \end{matrix} \right\|$$

(15) *Singular Systems.* Here $r < n$, and we have no difficulty if we use §17 (5), giving arbitrary (or particular) values to $n - r$ of the unknowns, and solve for the others in terms of them. Any nonzero $r$th-order minor of the matrix can now be taken as a basis for the solution of the set. When the equations are taken that involve this minor, and the superfluous unknowns are given particular values, we have a nonhomogeneous set of equations, the complementary functions of which can be obtained as in the foregoing theory. In addition, it will require a set of particular integrals determined as in Theorem III (§25).

(16) *Example:*

$$\left\| \begin{matrix} 1 & D & -D^2 \\ -D & 1 & D \\ 1 - D & D + 1 & D - D^2 \end{matrix} \right\| \equiv \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \right\|$$

$$\equiv (1 \mid 1 \mid 0) \qquad (I_2 \mid 0_1)$$

Choose rows 1 and 2 and columns 1 and 2 for our nonzero matrix, together with $y_{i3} = k_i$, obtaining

$$\begin{matrix} 1 & D & 0 \\ -D & 1 & 0 \end{matrix}$$

*i.e.*,

$$\begin{matrix} y_{i1} - Dy_{i2} & 0 \\ -Dy_{i1} + y_{i2} & 0 \end{matrix}$$

the solutions of which are

$$\text{I.}\quad \begin{aligned} y_{11} &= C_{11} \cos t + C_{12} \sin t \\ y_{12} &= -C_{11} \sin t + C_{12} \cos t \\ y_{13} &= k_1 \end{aligned}$$

$$\text{II.}\quad \begin{aligned} y_{21} &= C_{21} \sin t - C_{22} \cos t \\ y_{22} &= C_{21} \cos t + C_{22} \sin t \\ y_{23} &= k_2 \end{aligned}$$

## §25. The Particular Integrals.

(1) **Theorem III: Particular values of the unknowns $y_j$ can be found for the equations (A) by the use of Cramer's rule.***

(2) For proof by the operational method, we assume (*A*):

$$\sum_{j=1}^{n} F_{ij}(D) \cdot y_j = X_i, \qquad i = 1, 2, \cdots, n$$

Multiply through by the cofactors $f_{ik}(D)$ of the $F_{ij}(D)$ of any $k$th column:

$$f_{ik}(D) \cdot \sum_{j=1} F_{ij}(D) \cdot y_j = f_{ik}(D) \cdot X_i, \qquad i = 1, 2, \cdots, n$$

for any $k$

Add for every $i$, and regroup:

$$\sum_{j=1}^{n} \left[ \sum_{i=1}^{n} f_{ik}(D) \cdot F_{ij}(D) \right] \cdot y_j = \sum_{i=1}^{n} f_{ik}(D) \cdot X_i, \qquad k = 1, 2, \cdots, n$$

But

$$\left[ \sum_{i=1}^{n} f_{ik}(D) \cdot F_{ij}(D) \right] = | \; F_{ij}(D) \qquad \text{for} \quad k = j$$

$$= 0, \qquad \text{for} \quad k \neq j$$

and

$$\sum_{i=1} f_{ik}(D) \cdot X_i = | \; K_j \; |, \qquad \text{for} \quad k = j$$

where $| \; K_j \; |$ is the determinant $| \; F_{ij}(D) \; |$ with the $j$th column of $F_{ij}(D)$ replaced by the column of $X_i$. We thus have

$$| \; F_{ij}(D) \; | \cdot y_j = | \; K_j \; |$$

from which

$$y_j = | \; F_{ij}(D) \; |^{-1} \cdot | \; K_j \; |, \qquad j = 1, 2, \cdots, n \qquad (M)$$

(3) *The General Solution.* The integrals obtained by (*M*) complete the general solution of the set (*A*). Thus the general solution is the sum of (*D*) [or (*K*) and (*L*)] and (*M*).

* Bôcher, *op. cit.*, p. 43.

(4) In the singular case ($r < n$) for the nonhomogeneous case, it still remains nonhomogeneous upon giving arbitrary values to $n - r$ of the unknowns, and the method of §24 (15) is used for its solution with the addition of ($M$).

(5) The theorem that the augmented matrix must have the same rank as that of the matrix itself does not apply, for the $X_i$ are functions of the independent variable, and the $F_{ij}(D)$ are functions of $D$. As a matter of fact, both always have the same rank.

(6) As an example of Theorem III, we shall use the matrix

$$\begin{matrix} D^2 & -2a\omega D & 0 & 0 \\ 2a\omega D & D^2 & 2b\omega D & 0 \\ 0 & -2b\omega D & D^2 & -g \end{matrix}$$

the complementary functions of which were found by Theorems I and II under §24 (13). Apply Theorem III.

$$\begin{aligned} \Delta &\equiv D^4(D^2 + 4\omega^2) \\ f_{31} &\equiv -4ab\omega^2 D^2 \\ f_{32} &\equiv -2b\omega D^3 \\ f_{33} &\equiv D^2(D^2 + 4a^2\omega^2) \end{aligned}$$

Then

$$\begin{aligned} \bar{y}_{i1} &= \frac{1}{D^4(D^2 + 4\omega^2)}(-4ab\omega^2 D^2)(-g) \\ \bar{y}_{i2} &= \frac{1}{D^4(D^2 + 4\omega^2)}(-2b\omega D^3)(-g) \\ \bar{y}_{i3} &= \frac{1}{D^4(D^2 + 4\omega^2)}[D^2(D^2 + 4a^2\omega^2)](-g) \end{aligned}$$

Simplifying the operators, and operating,

$$\begin{aligned} \bar{y}_{i1} &= \frac{4ab\omega^2 g}{D^2(D^2 + 4\omega^2)} \cdot 1 = abg\frac{t^2}{2} \\ \bar{y}_{i2} &= \frac{2b\omega g}{D(D^2 + 4\omega^2)} \cdot 1 = \frac{bg}{2\omega}t \\ \bar{y}_{i3} &= \frac{-g}{D^2} \cdot 1 = -\frac{gt^2}{2} \end{aligned}$$

These should be added to the complementary functions of each of the three fundamental sets previously found, and then we have

the complete or general solution of the nonhomogeneous set of the present section.

**§26. Examples for Chap. VI:**

(1) $A\frac{dp}{dt} = -(C - A)nq$ [Lamb]

$$A\frac{dq}{dt} = (C - A)np$$

(2) $\frac{dx}{dt} = \pm\omega y$ [Piaggio]

$$\frac{dy}{dt} \qquad \omega x$$

(3) $\frac{d^2x}{dt^2} = -n^2x$ [Lamb]

$$\frac{d^2y}{dt^2} = -n^2y$$

(4) $\frac{d^2x}{dt^2} + 2k\frac{dy}{dt} + c^2x = 0$ [Gray]

$$\frac{d^2y}{dt^2} - 2k\frac{dx}{dt} + c^2y = 0$$

(5) $A\frac{d^2x}{dt^2} + C\omega\frac{dy}{dt} + Mghx = 0$ [Lamb]

$$A\frac{d^2y}{dt^2} - C\omega\frac{dx}{dt} + Mghy = 0$$

(6) $m\frac{d^2x}{dt^2} = -P\frac{x}{a} + P\left(\frac{y - x}{2b}\right)$ [Lamb]

$$m\frac{d^2y}{dt^2} = -P\frac{y}{a} - P\left(\frac{y - x}{2b}\right)$$

(7) $\frac{dP_1}{dt} = -\lambda_1P_1$ [Bateman]

$$\frac{dP_2}{dt} = \lambda_1P_1 - \lambda_2P_2$$

$$\frac{dP_3}{dt} = \lambda_2P_2$$

(8) $\frac{dx}{dt} = ry - qz$ [Lamb]

$$\frac{dy}{dt} = pz - rx$$

$$\frac{dz}{dt} = qx - py$$

(9) $m\dfrac{d^2y_1}{dt^2} + \dfrac{P}{a}(2y_1 - y_2) = 0$ [Lamb]

$m\dfrac{d^2y_2}{dt^2} + \dfrac{P}{a}(2y_2 - y_1 - y_3) = 0$

$m\dfrac{d^2y_3}{dt^2} + \dfrac{P}{a}(2y_3 - y_2) = 0$

(10) $\dfrac{dP_0}{dt} = -\lambda_0 P_0$ [Bateman]

$\dfrac{dP_k}{dt} = \lambda_{k-1}P_{k-1} - \lambda_k P_k \qquad k = 2, 3, \cdots, n$

(11) $\dfrac{d^2x}{dt^2} = 0$ [Lamb]

$\dfrac{d^2y}{dt^2} = -g$

(12) $\left\| \begin{matrix} D^2 & -2a\omega D & 0 & 0 \\ 2a\omega D & D^2 & 2b\omega D & 0 \\ 0 & 2b\omega D & D^2 & -g \end{matrix} \right\|$ [Appell]

$a^2 + b^2 = 1$

Relative motion under gravity alone.

Unrestricted flight of a particle.

$$\left.\begin{matrix} x = 0, & y = 0, & z = 0 \\ \dfrac{dx}{dt} = u_0, & \dfrac{dy}{dt} = v_0, & \dfrac{dz}{dt} = w_0 \end{matrix}\right\} t = 0$$

(13) $\dfrac{dx}{dt} + ax = 0$ [Piaggio]

$\dfrac{dz}{dt} = by$

$x + y + z = C$

(14) $(D + 1)x + Dy = 0$ [Chrystal]

$(D + 3)x + (D + 2)y = 0$

(15) $\left\| \begin{matrix} D - 1 & 1 \\ D & -2 \end{matrix} \right\|$ [Phillips]

(16) $\left\| \begin{matrix} 7 & D - 3 \\ 7D + 63 & -36 \end{matrix} \right\|$ [Campbell]

(17) $\left\| \begin{matrix} D - 3 & 1 \\ -1 & D - 1 \end{matrix} \right\|$ [Mellor]

$$(18)\quad \left\| \begin{matrix} D - a_{11} & a_{12} \\ a_{21} & D - a_{22} \end{matrix} \right\| \qquad \text{[Moulton]}$$

$$(19)\quad \left\| \begin{matrix} D^2 & m^2 \\ -m^2 & D^2 \end{matrix} \right\| \qquad \text{[Murray]}$$

$$(20)\quad \left\| \begin{matrix} -1 & D^2 \\ D^2 & -1 \end{matrix} \right\| \qquad \text{[Phillips]}$$

$$(21)\quad \left\| \begin{matrix} D^3 + 2D^2 + D + 1 & D^3 + 2D + 1 \\ D^2 + 2D + 1 & D^4 + D + 2 \end{matrix} \right\| \qquad \text{[Routh]}$$

$$(22)\quad \left\| \begin{matrix} D^2 - 3D + 2 & D - 1 \\ -(D - 1) & D^2 - 5D + 4 \end{matrix} \right\| \qquad \text{[Routh]}$$

$$(23)\quad \left\| \begin{matrix} D^2 + 1 & D^2 + D + 1 & t \\ D & D + 1 & e^t \end{matrix} \right\| \qquad \text{[Chrystal]}$$

$$(24)\quad \left\| \begin{matrix} 4D + 3 & -D & \sin t \\ D & 1 & \cos t \end{matrix} \right\| \qquad \text{[Phillips]}$$

$$(25)\quad \left\| \begin{matrix} D - 1 & 1 & t - 2 \\ 0 & D - 1 & t \end{matrix} \right\| \qquad x = y = t = 0 \quad \text{[Bateman]}$$

$$(26)\quad \left\| \begin{matrix} D - a & -b & f(t) \\ -l & D - m & g(t) \end{matrix} \right\| \qquad \text{[Bateman]}$$

$$(27)\quad \left\| \begin{matrix} D^2 - 3 & -1 & e^t \\ -2 & D & 0 \end{matrix} \right\| \qquad \text{[Phillips]}$$

$$(28)\quad \left\| \begin{matrix} D^2 + 1 & -D - 1 & \cos nt \\ 2D & D + 1 & 0 \end{matrix} \right\| \qquad \text{[Fry]}$$

$$(29)\quad \left\| \begin{matrix} D^2 & nD & a \cos nt \\ -nD & D^2 & 0 \end{matrix} \right\| \qquad \text{[Ince]}$$

$$(30)\quad \left\| \begin{matrix} -D & D^2 & -a \\ D^2 & D & \theta \end{matrix} \right\| \qquad \text{[Bateman]}$$

$$(31)\quad \left\| \begin{matrix} D - a & 0 & 0 \\ -1 & D - a & 0 \\ 0 & -1 & D - a \end{matrix} \right\| \qquad \text{[Piaggio]}$$

$$(32)\quad \left\| \begin{matrix} D^2 - 1 & 2D + 2 & D + 1 & 2e^t \\ (D - 1)^2 & 4D & D - 3 & 0 \\ 3D - D^3 & -2D & -D + 1 & 0 \end{matrix} \right\| \qquad \text{[Ince]}$$

$$(33)\quad \left\| \begin{matrix} D^2 - \omega^2 & -2\omega D & 0 & 0 \\ 2\omega D & D^2 - \omega^2 & 0 & 0 \\ 0 & 0 & D^2 & -g \end{matrix} \right\| \qquad \text{[Piaggio]}$$

(34) $$\begin{vmatrix} 2D-5 & 1 & 2 & 0 \\ 1 & 2D-5 & 0 & 2 \\ 2 & 0 & 2D-5 & 1 \\ 0 & 2 & 1 & 2D-5 \end{vmatrix}$$ [Moulton]

(35) $$\begin{aligned} \ddot{D}X &= k_1(a - X) \\ Du &= k_2(x - u) \\ Dv &= k_3(y - v) \\ v &= k_4 u \\ X &= x + y + u + v \\ u &= v = x = y = t = 0 \end{aligned}$$ [Mellor]

(36) $$\frac{dy_i}{dx} + \sum_j a_{ij} y_j = 0 \qquad i = 1, 2 \cdot \qquad n$$ [Goursat]

(37) $$\begin{aligned} M\frac{dC_1}{dt} + L_2\frac{dC_2}{dt} + R_2C_2 &= E_2 \\ M\frac{dC_2}{dt} + L_1\frac{dC_1}{dt} + R_1C_1 &= E_1 \end{aligned}$$ [Mellor]

(38) $$\begin{aligned} L_1D^2I_1 + MD^2I_2 + \frac{1}{C_1}I_1 &= Ep\cos pt \\ L_2D^2I_2 + MD^2I_1 + \frac{1}{C_2}I_2 &= 0 \end{aligned}$$ [Piaggio]

(39) Use (37) with [Gray]

$$\begin{aligned} E_1 &= \text{constant}, \qquad L_1 = 0.613 \text{ henry} = L_2 \qquad M \\ E_2 &= 0, \qquad R_1 = 12.3 \text{ ohms} = R_2 \end{aligned}$$

(40) $$\begin{aligned} R_1I_1 + L_1\frac{dI_1}{dt} + M\frac{dI_2}{dt} + \frac{1}{C_1}\int I_1 dt &= E \\ R_2I_2 + L_2\frac{dI_2}{dt} + M\frac{dI_1}{dt} + \frac{1}{C_2}\int I_2 dt &= 0 \end{aligned}$$ [Cohen]

(41) $$\begin{aligned} R_1I_1 + L_1\frac{dI_1}{dt} + L_2\frac{dI_2}{dt} + R_2I_2 &= E_1 \\ R_2I_2 + L_2\frac{dI_2}{dt} + L_3\frac{dI_3}{dt} + R_3I_3 &= E_2 \\ R_3I_3 + L_3\frac{dI_3}{dt} + L_1\frac{dI_1}{dt} + R_1I_1 &= E_3 \\ E_1 = E_2 = E_3 &= 120 \text{ volts} \\ R_1 = R_2 = R_3 &= 9 \text{ ohms} \\ L_1 = L_2 = L_3 &= 0.205 \text{ henry} \end{aligned}$$ [Magnussen]

(42) $$\left\|\begin{array}{cc|c} mD^2 & HeD & Ve \\ -HeD & mD^2 & 0 \end{array}\right\|$$ [Piaggio]

$$x = 0, \qquad y = 0, \qquad t = 0$$

$$\frac{dx}{dt} = 0, \qquad \frac{dy}{dt} = 0, \qquad t = 0$$

(43) $$\begin{aligned} &(LD + 2R)I_1 - RI_2 = E \\ &-RI_1 + (LD + 2R)I_2 - RI_3 = 0 \\ &-RI_2 + (LD + 2R)I_3 - RI_4 = 0 \\ &-RI_3 + (LD + 2R)I_4 = 0 \end{aligned}$$

# CHAPTER VII

## THE OPERATORS $d_1 \equiv \frac{\partial}{\partial x}$, $d_2 \equiv \frac{\partial}{\partial y}$

**§27. The Problem.**

We shall now extend our operational theory to three-dimensional geometry and physical problems, where two variables are independent, and one dependent. To distinguish between the two independent variables, we use the subscript notation and develop the algebra of partial operators, which follows very closely that of the single ordinary operator.

**§28. Definitions and Elementary Algebra.**

(1) With $d_i \equiv \frac{\partial}{\partial x_i}$, we have

$$d_1 \equiv \frac{\partial}{\partial x_1} \equiv \frac{\partial}{\partial x}, \qquad d_2 \equiv \frac{\partial}{\partial x_2} \equiv \frac{\partial}{\partial y}$$

(2) Then we have the inverses

$$d_1^{-1} \equiv \frac{1}{d_1} \equiv \int^x ()\partial x + f(y), \qquad \text{or} \qquad \int_a^x ()\partial x$$

$$d_2^{-1} \equiv \frac{1}{d_2} \equiv \int^y ()\partial y + \phi(x), \qquad \text{or} \qquad \int_a^y ()\partial y$$

(3) *Iteration.* As with the ordinary operator $D$, powers of $d$ mean repeated or iterated differentiation or integration; thus,

$$d_1^n \equiv \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial x} \qquad \frac{\partial}{\partial x} \quad \frac{\partial^n}{\partial x^n}$$

$$d_2^n \equiv \frac{\partial}{\partial y} \cdot \frac{\partial}{\partial y} \cdots \frac{\partial}{\partial y} \equiv \frac{\partial^n}{\partial y^n}$$

$$d_1^m d_2^n \equiv \frac{\partial}{\partial x} \qquad \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial y}$$

and

$$d_1^{-n} \equiv \int^x \cdots \int^x ()\partial x^n + E(y) \equiv \int_a^x \cdots \int_a^x ()\partial x^n$$

$$d_2^{-n} \equiv \int^y \cdot \cdot \cdot \int^y ()\partial y^n + F(x) \equiv \int_a^y \cdot \cdot \cdot \int_a^y ()\partial y^n$$

etc.

(4) *Distribution.* These operators, powers of them, and any rational integral functions of them are distributive operators. Included in the latter are also any functions that can be expanded by Taylor's theorem or by actual algebraic operations into infinite series of ascending or descending powers of the operator.

(5) *Commutation.* The partial operators are commutative, with respect to both addition and multiplication, with themselves and constants; *i.e.*,

$$\begin{aligned} d_1d_2 &\equiv d_2d_1 \\ d_id_j &\equiv d_jd_i \\ \pi d_ic &\equiv c\pi d_i \\ d_1d_2(u+v) &\equiv d_1d_2(v+u) \\ f(d_1, d_2) : \phi(d_1, d_2) &\equiv \phi(d_1, d_2) \cdot f(d_1, d_2) \\ (d_1 - a)(d_2 - b) &\equiv (d_2 - b)(d_1 - a) \end{aligned}$$

But it must be noticed that we cannot have functions of $d_1$, $d_2$, $x$, and $y$ commutative. Thus,

$$\begin{aligned} (d_1 - x)(d_2 - x) &\not\equiv (d_2 - x)(d_1 - x) \\ (d_1 - y^2)(d_2 - 2y) &\not\equiv (d_2 - 2y)(d_1 - y^2) \end{aligned}$$

etc.

(6) *Index Law.* For positive integral indices, we can have

$$d_i^m d_i^n \equiv d_i^{m+n}$$

and

$$[f(d_1, d_2)]^m \cdot [f(d_1, d_2)]^n \equiv [f(d_1, d_2)]^{m+n}$$

(7) For negative integral indices, if we neglect the appendage, we have

$$d_i^m d_i^{-n} \equiv d_i^{-n} d_i^m \equiv d_i^{m-n}$$

for which, if $m = n$,

$$d_i^m d_i^{-m} \equiv d_i^{-m} d_i^m \equiv d_i^0 \equiv 1$$

and

$$f^m \cdot f^{-m} \equiv f^{-m} \cdot f^m \equiv 1$$

where $f \equiv f(d_1, d_2, c)$.

(8) Partial operators of inverse type can be expanded into ascending or descending series in the differentiator $d_i$ in the same manner as was indicated for the operator $D$ [II, §4 (12–15)].

(9) However, since the operators here may contain $d_1$ and $d_2$, we should add that the series may be ascending or descending in either $d_1$ or $d_2$ or both, according to the form to be expanded. Illustrations of the use of the expansions will be found later.

(10) Expand the following:

(a)

$$\begin{aligned}
\frac{1}{d_1 - d_2} &= \frac{1}{d_1 \cdot \left(1 - \frac{d_2}{d_1}\right)} \\
&\equiv \left(1 - \frac{d_2}{d_1}\right)^{-1} \cdot \frac{1}{d_1} \\
&\equiv \left[1 + \frac{d_2}{d_1} + \frac{d_2^2}{d_1^2} + \frac{d_2^3}{d_1^3} + \qquad \right] \cdot \frac{1}{d_1} \\
&\equiv \frac{1}{d_1} + \frac{d_2}{d_1^2} + \frac{d_2^2}{d_1^3} + \frac{d_2^3}{d_1^4} +
\end{aligned}$$

or

$$\begin{aligned}
\frac{-1}{d_2 \cdot \left(1 - \frac{d_1}{d_2}\right)} &\equiv -\left(1 - \frac{d_1}{d_2}\right)^{-1} \cdot \frac{1}{d_2} \\
&\equiv -\left[1 + \frac{d_1}{d_2} + \frac{d_1^2}{d_2^2} + \frac{d_1^3}{d_2^3} + \cdot \qquad \right] \\
&\equiv -\left[\frac{1}{d_2} + \frac{d_1}{d_2^2} + \frac{d_1^2}{d_2^3} + \frac{d_1^3}{d_2^4} + \cdots \right]
\end{aligned}$$

(b) $\dfrac{1}{d_1 - ad_2}$ (d) $\dfrac{1}{(d_1 - d_2)(d_1 - 2d_2)}$ (f) $\dfrac{1}{(d_1 - 1)(d_2 - 1)}$

(c) $\dfrac{1}{2d_1 - 3d_2}$ (e) $\dfrac{1}{(d_1 - d_2)^2}$ (g) $\dfrac{1}{d_1 - d_2 - 1}$

(11) *Partial Fractions.* The inverse partial operators can be expanded into partial fractions, as with ordinary operators.

Use the same rules as in II §4 (18). A résumé of the possible forms here will be useful, inasmuch as there will be additional operators present in each partial fraction.

Type I. $F(d_1, d_2) \equiv (d_1 - \alpha d_2)f(d_1, d_2)$ gives a partial fraction of the form

$$\frac{1}{d_2 \cdot f(\alpha, 1)(d_1 - \alpha d_2)}$$

Type II. $F(d_1, d_2) \equiv (d_1 - \alpha d_2)^p \phi(d_1, d_2)$ gives a partial fraction series of the form

$$\sum_{i=1}^{p} \frac{N_i(\alpha)}{d_2^{k-i} \cdot (d_1 - \alpha d_2)^i}, \qquad k \text{ being the degree of } F \text{ in } d_2$$

Type III. $F(d_1, d_2) \equiv [(d_1 - \alpha d_2)^2 + \beta^2 d_2^2] \cdot f(d_1, d_2)$ gives a partial fraction of the form

$$\frac{Ld_1 + Md_2}{d_2^{k-2}[(d_1 - \alpha d_2)^2 + \beta^2 d_2^2]}$$

Type IV. $F(d_1, d_2) \equiv [(d_1 - \alpha d_2)^2 + \beta_2 d_2^2]^p \cdot \phi(d_1, d_2)$ gives form

$$\sum_{i=1}^{p} \frac{L_i d_1 + M_i d_2}{d_2^{k-i}[(d_1 - \alpha d_2)^2 + \beta^2 d_2^2]^i}$$

(12) The student should expand the following into partial fractions:

(a) $\dfrac{1}{d_1^2 + d_1 d_2}$

(b) $\dfrac{1}{d_1^2 - d_2^2}$

(c) $\dfrac{1}{d_1^2 - a^2 d_2^2}$

(d) $\dfrac{1}{d_1^2 + 2d_1 d_2 + d_2^2}$

(e) $\dfrac{1}{d_1^2 - 4d_1 d_2 + 4d_2^2}$

(f) $\dfrac{1}{(d_1^2 + d_2^2)(d_1 - d_2)}$

(g) $\dfrac{1}{2d_1^2 - 3d_1 d_2 - 2d_2^2}$

(h) $\dfrac{1}{d_1^3 + d_1^2 d_2 - d_1 d_2^2 - d_2^3}$

(i) $\dfrac{1}{d_1^3 - 3d_1^2 d_2 + 2d_1 d_2^2}$

(j) $\dfrac{1}{d_1^2 - d_2^2 - d_1 + d_2}$

(k) $\dfrac{1}{d_1 - a^2 d_2^2}$

(l) $\dfrac{1}{d_1^2 - d_2}$

(m) $\dfrac{1}{d_1 + d_2^2}$ (n) $\dfrac{1}{(d_1 - d_2)^2(d_1 + d_2)}$

(o) $\dfrac{1}{(d_1 - 2d_2)^2(d_1 - d_2)(d_1 - 3d_2)}$

**§29. Fundamental Theorems.**

(1) The fundamental theorems here shown for functions of two partial operators are somewhat similar to those for the ordinary operator, but the differences are noteworthy. The principal theorems are the following:

$$\text{I. } F(d_1, d_2) \cdot e^{\phi(x,y)} \equiv e^{\phi(x,y)} \cdot F\left[d_1 + \frac{\partial \phi}{\partial x}, d_2 + \frac{\partial \phi}{\partial y}\right]$$

$$\text{II. } e^{\phi(d_1,d_2)} \cdot F(x, y) \equiv F\left[x + \frac{\partial \phi}{\partial d_1}, y + \frac{\partial \phi}{\partial d_2}\right] \cdot e^{\phi(d_1,d_2)}$$

$$\text{III. } F(d_1, d_2) \cdot \phi(ax + by) \equiv F(a, b) \cdot \phi^{(n)}(ax + by)$$

$$\text{IV. } F(d_1, d_2) \equiv e^{p_1\frac{\partial}{\partial q_1} + p_2\frac{\partial}{\partial q_2}} \cdot F(q_1, q_2)$$

$$\equiv \sum_{k=0}^{\infty} \frac{1}{k!}\left[p_1\frac{\partial}{\partial q_1} + p_2\frac{\partial}{\partial q_2}\right]^k \cdot F(q_1, q_2)$$

where $d_1 = p_1 + q_1$, $d_2 = p_2 + q_2$, $p_1$, $p_2 \sim P(x, y)$, $q_1$, $q_2 \sim Q(x, y)$, $F \sim P \cdot Q$.

(2) *Theorem* I. This theorem was first derived in 1837 by Robert Murphy [*cf.* Appendix III, §60 (12)], later in 1853 in a distinct manner by Charles Graves [*cf.* Appendix III, §62 (12)], both operationally. The proof by induction follows that for the ordinary operator as shown in Chap. II [§5 (2)], outlined as follows:

Since

$$d_1 \cdot e^{\phi(x,y)}S = e^{\phi(x,y)}d_1S + S \cdot d_1e^{\phi(x,y)}$$

$$= e^{\phi(x,y)}d_1S + e^{\phi(x,y)} \cdot \frac{\partial \phi}{\partial x}S$$

$$= e^{\phi(x,y)}\left[d_1 + \frac{\partial \phi}{\partial x}\right]S$$

we have

$$d_1 \cdot e^{\phi(x,y)} \equiv e^{\phi(x,y)} \cdot \left(d_1 + \frac{\partial \phi}{\partial x}\right)$$

Operate on the left of this by $d_1$:

$$\begin{aligned} d_1{}^2 e^{\phi(x,y)} &\equiv [d_1 e^{\phi(x,y)}]\left(d_1 + \frac{\partial \phi}{\partial x}\right) \\ &\equiv \left[e^{\phi(x,y)}\left(d_1 + \frac{\partial \phi}{\partial x}\right)\right]\left(d_1 + \frac{\partial \phi}{\partial x}\right) \\ &\equiv e^{\phi(x,y)}\left(d_1 + \frac{\partial \phi}{\partial x}\right)^2 \end{aligned}$$

and so on, till

$$d_1{}^n e^{\phi(x,y)} \equiv e^{\phi(x,y)}\left(d_1 + \frac{\partial \phi}{\partial x}\right)^n$$

By the same method, operate successively by $d_2$ on this form, obtaining

$$d_2{}^m d_1{}^n e^{\phi(x,y)} \equiv e^{\phi(x,y)}\left(d_2 + \frac{\partial \phi}{\partial y}\right)^m \left(d_1 + \frac{\partial \phi}{\partial x}\right)^n$$

from which a rational integral function can be built up of the left-hand side in $d_1$ and $d_2$ and of the right-hand side in $d_1 + \dfrac{\partial \phi}{\partial x}$ and $d_2 + \dfrac{\partial \phi}{\partial y}$; *i.e.*

$$F(d_1,\ d_2)e^{\phi(x,y)} \equiv e^{\phi(x,y)} F\left[d_1 + \frac{\partial \phi}{\partial x},\ d_2 + \frac{\partial \phi}{\partial y}\right] \tag{I}$$

This theorem might be called the "shifting theorem" for two variables.

(3) For the inverse, use (I), and operate on its left by $F^{-1}(d_1,\ d_2)$ and on its right by $F^{-1}\left[d_1 + \dfrac{\partial \phi}{\partial x},\ d_2 + \dfrac{\partial \phi}{\partial y}\right]$. Then, with $F^{-1} \cdot F \equiv F \cdot F^{-1} \equiv 1$, ignoring the appendage, we shall have

$$e^{\phi(x,y)} F^{-1}\left[d_1 + \frac{\partial \phi}{\partial x},\ d_2 + \frac{\partial \phi}{\partial y}\right] \equiv F^{-1}(d_1,\ d_2)e^{\phi(x,y)} \tag{Ia}$$

(4) With $\phi(x, y) \equiv ax + by$, we have

$$\frac{\partial \phi}{\partial x} = a, \qquad \frac{\partial \phi}{\partial y} = b$$

and Theorem (I) becomes

$$F(d_1, d_2)e^{ax+by} \equiv e^{ax+by}F(d_1 + a, d_2 + b) \tag{Ib}$$

(5) Then, with a subject $S \equiv 1$, and

$$F(d_1 + a, d_2 + b) \equiv F(a, b) + d_1 \cdot F_a'(a, b) + d_2 \cdot F_b'(a, b) + \tfrac{1}{2}[F_a''(a, b) + 2F_{ab}''(a, b) + F_b''(a, b)] + \cdots$$

we shall have

$$\begin{aligned} F(d_1, d_2) \cdot e^{ax+by} \cdot 1 &= e^{ax+by}F[d_1 + a, d_2 + b] \cdot 1 \\ &= e^{ax+by} \cdot F(a, b) \end{aligned} \tag{Ic}$$

(6) If the inverse theorem is used in this manner, there must be a proviso that $F(a, b) \neq 0$. In case $F(a, b) = 0$, see Theorem III, (15) below.

(7) *Examples.* Find identical operators for each, and operate by them on the subject $S \equiv 1$.

(a) $(d_1 + d_2)e^{x^2y}$

(b) $\dfrac{1}{d_1 + d_2}e^{2x+3y}$

(c) $d_1^2d_2^3e^{x^2+y^2}$

(d) $\dfrac{1}{d_1d_2}e^{x^2+y^2}$

(e) $(d_1^2 + d_2^2)e^{x+y}$

(f) $\dfrac{1}{d_1^2 + d_2^2}e^{x-y}$

(g) $[\sin (d_1 + d_2)]e^{2x+y}$

(8) *Theorem* II. This theorem is called the correlative of Theorem I, inasmuch as it can be obtained from it by the use of Charles Graves's correlative theorem: If

$$\phi(\pi, \rho) = 0$$

then

$$\phi(\rho, -\pi) = 0$$

given in Appendix III [§62 (2)], and his general theorem

$$f\left[\pi_1 + \frac{\partial \phi}{\partial \rho_1}, \pi_2 + \frac{\partial \phi}{\partial \rho_2}\right] \equiv e^{\phi(\rho_1,\rho_2)} \cdot f(\pi_1, \pi_2) \cdot e^{-\phi(\rho_1,\rho_2)}$$

as stated in Appendix III [§62 (12)].

(9) It can, however, be derived very simply, as follows; using Leibnitz's extension,

$$F(d_1)xS = xF(d_1)S + (d_1x) \cdot F'(d_1)S$$

Then with

$$F(d_1) \equiv e^{-\phi(d_1,d_2)}$$
$$S \equiv e^{\phi(d_1,d_2)}S'$$

we have

$$e^{-\phi(d_1,d_2)}xe^{\phi(d_1,d_2)}S' = xe^{-\phi(d_1,d_2)}e^{\phi(d_1,d_2)}S' + \left(-\frac{\partial\phi}{\partial d_1}\right)e^{-\phi(d_1,d_2)}e^{\phi(d_1,d_2)}S$$
$$= \left(x - \frac{\partial\phi}{\partial d_1}\right)S'$$

Abstracting the operators,

$$e^{-\phi}\cdot x\cdot e^{\phi} \equiv x - \frac{\partial\phi}{\partial d_1}$$

Now iterate this operator

$$[e^{-\phi}xe^{\phi}]^n \equiv \left(x - \frac{\partial\phi}{\partial d_1}\right)^n$$

the left-hand side of which extended is

$$e^{-\phi}xe^{\phi}\cdot e^{-\phi}xe^{\phi}\cdot e^{-\phi}xe^{\phi}\ \cdots\ e^{-\phi}xe^{\phi} \equiv e^{-\phi}x^ne^{\phi}$$

giving us

$$e^{-\phi}x^ne^{\phi} \equiv \left(x - \frac{\partial\phi}{\partial d_1}\right)^n, \qquad \text{by} \qquad e^{\phi}e^{-\phi} \equiv 1$$

Similarly, we can obtain

$$e^{-\phi}y^me^{\phi} \equiv \left(y - \frac{\partial\phi}{\partial d_2}\right)^n$$

Compound these:

$$e^{-\phi}x^ne^{\phi}\cdot e^{-\phi}y^me^{\phi} \equiv \left(x - \frac{\partial\phi}{\partial d_1}\right)^n\cdot\left(y - \frac{\partial\phi}{\partial d_2}\right.$$

giving

$$e^{-\phi}x^ny^me^{\phi} \quad \left(x - \frac{\partial\phi}{\partial d_1}\right)^n\cdot\left(y - \frac{\partial\phi}{\partial d_2}\right)$$

Now form a rational integral function

$$\sum_{n,m} C_{nm} e^{-\phi} x^n y^m e^{\phi} \equiv \sum_{n,m} C_{nm}\left(x - \frac{\partial\phi}{\partial d_1}\right)^n \left(y - \frac{\partial\phi}{\partial d_2}\right)^m$$

or

$$e^{-\phi}\left[\sum_{n,m} C_{nm} x^n y^m\right] e^{\phi} \equiv \sum_{n,m} C_{nm}\left(x - \frac{\partial\phi}{\partial d_1}\right)^n \left(y - \frac{\partial\phi}{\partial d_2}\right)^m$$

*i.e.*,

$$e^{-\phi} F(x, y) e^{\phi} \equiv F\left[x - \frac{\partial\phi}{\partial d_1}, y - \frac{\partial\phi}{\partial d_2}\right]$$

Operate on the right by $e^{-\phi}$; then put $-\phi$ for $\phi$, obtaining

$$e^{\phi(d_1,d_2)} \cdot F(x, y) \equiv F\left[x + \frac{\partial\phi}{\partial d_1}, y + \frac{\partial\phi}{\partial d_2}\right] \cdot e^{\phi(d_1,d_2)} \qquad \text{(II)}$$

the theorem desired.

(10) For the inverse of this, proceed as for the inverse of (I), obtaining

$$e^{\phi(d_1,d_2)} F^{-1}(x, y) \equiv F^{-1}\left[x + \frac{\partial\phi}{\partial d_1}, y + \frac{\partial\phi}{\partial d_2}\right] e^{\phi(d_1,d_2)} \qquad \text{(II}a\text{)}$$

(11) Using $\phi(d_1, d_2) \equiv hd_1 + kd_2$, (II) becomes

$$e^{hd_1+kd_2} F(x, y) \equiv F(x + h, y + k) e^{hd_1+kd_2} \qquad \text{(II}b\text{)}$$

(12) Apply (II$b$) to a subject $S \equiv 1$.

$$e^{hd_1+kd_2} F(x, y) 1 = F(x + h, y + k) e^{hd_1+kd_2} 1$$

but

$$e^{hd_1+kd_2} 1 = [1 + hd_1 + kd_2 + \cdots ] 1 = 1$$

so that

$$e^{hd_1+kd_2} F(x, y) = F(x + h, y + k) \qquad \text{(II}c\text{)}$$

This is Taylor's theorem for two variables, in symbolical form.

(13) If $e^h = a$, $e^k = b$, we shall have variations of (II$b$) and (II$c$):

$$a^{d_1} b^{d_2} F(x, y) \equiv F(x + \log a, y + \log b) a^{d_1} b^{d_2} \qquad \text{(II}d\text{)}$$

$$a^{d_1} b^{d_2} F(x, y) = F(x + \log a, y + \log b) \qquad \text{(II}e\text{)}$$

(14) Theorem II is particularly useful in obtaining the complementary functions for partial linear differential equations with constant coefficients and will be exemplified in that connection later.

(15) *Theorem* III. This theorem can be proved by induction, as follows: Set

$$t = ax + by$$

Then

$$\begin{aligned} d_1{}^n\phi(t) &= \frac{\partial^n}{\partial t^n}\phi(t) \cdot \left(\frac{\partial t}{\partial x}\right)^n = a^n\phi^{(n)}(ax + by) \\ d_2{}^m\phi(t) &= \frac{\partial^m}{\partial t^m}\phi(t) \cdot \left(\frac{\partial t}{\partial y}\right)^m = b^m\phi^{(m)}(ax + by) \\ d_1{}^n d_2{}^m\phi(t) &= a^n b^m \phi^{(m+n)}(ax + by) \end{aligned}$$

Form a homogeneous rational function of degree $n$ in $d_1$, $d_2$, and obtain the theorem.

(16) For the inverse, use

$$F(d_1, d_2)\phi(ax + by) = F(a, b)\phi^{(n)}(ax + by)$$

and operate on the left by $F^{-1}(d_1, d_2)$ and on the right by $F^{-1}(a, b)$; also, using $F^{-1}F \equiv FF^{-1} \equiv 1$; we then have

$$F^{-1}(d_1, d_2)\phi^{(n)}(ax + by) = F^{-1}(a, b)\phi(ax + by)$$

Call $\phi^{(n)} \equiv \Phi$, and obtain

$$F^{-1}(d_1, d_2)\Phi(ax + by) = F^{-1}(a, b)\Phi^{(-n)}(ax + by) \qquad \text{(III}a\text{)}$$

where $(-n)$ indicates $n$ iterated integrations of $\Phi$ with respect to $t = ax + by$.

(17) There is an exceptional case to the inverse, *i.e.*, where the form $F(a, b) = 0$. In order to operate properly then, $F(d_1, d_2)$ should be factored, and the factor causing the zero isolated and used separately. This will be shown in detail under §30 ("Elementary Interpretation.")

(18) *Examples:*

(a) $(d_1 + d_2)(x + y)$

(b) $(d_1^2 + 2d_1d_2 + 3d_2^2)(x + y)^2$

(c) $\dfrac{1}{d_1 - d_2}(x - y)$

(d) $\dfrac{1}{(d_1 + d_2)^2}(2x - 3y)^{-2}$

(e) $(d_1^2 + d_2^2)\sin(2x - 3y)$

(f) $\dfrac{1}{d_1^2 + d_2^2}\cos(2x - 3y)$

(g) $(d_1^2 - 3d_1d_2 + 4d_2^2)\log(x + y)$

(h) $\dfrac{1}{(d_1 - d_2)(d_1 - 2d_2)}e^{3x+y}$

(i) $(d_1 - d_2)\sqrt{ax - by}$

(j) $\dfrac{1}{d_1 - d_2}\sqrt{ax - by}$

(k) $(d_1^2 - d_2^2)e^{(x-2y)^2}$

(l) $\dfrac{1}{d_1^2 - d_2^2}e^{(x-2y)^2}$

(19) *Theorem* IV. This is an extension of Theorem VI [II §5 (22)] to functions of two independent variables. Using Taylor's theorem for two variables as a basis,

$$e^{hd_1+kd_2}F(x, y) = F(x + h, y + k)$$

we make substitutions as follows:

$$\begin{aligned} x &= q_1, & h &= p_1, & x + h &= d_1 & \frac{\partial}{\partial x} &= \frac{\partial}{\partial q_1} \\ y &= q_2, & k &= p_2, & y + k &= d_2 = \frac{\partial}{\partial y} &&= \frac{\partial}{\partial q_2} \end{aligned}$$

$p_1, p_2$ operating only on $P(x, y)$
$q_1, q_2$ operating only on $Q(x, y)$

Then directly,

$$e^{p_1\frac{\partial}{\partial q_1} + p_2\frac{\partial}{\partial q_2}} \cdot f(q_1, q_2) \equiv f(d_1, d_2)$$

Expand the exponential:

$$f(d_1, d_2) \equiv \sum_{k=0}^{\infty}\frac{1}{k!}\left(p_1\frac{\partial}{\partial q_1} + p_2\frac{\partial}{\partial q_2}\right) \cdot f(q_1, q_2)$$

$$\equiv \sum_{n=0}\left[\sum_{k=0}^{n}\frac{1}{k!(n - k)!}p_1^k p_2^{n-k}\frac{\partial^n}{\partial q_1^k \partial q_2^{n-k}}f(q_1, q_2)\right]$$

(20) This theorem is useful when $f(d_1, d_2)$ is to operate on a product made up of two factors which are functions of $x$, $y$, one of which is a power function; thus,

$$P \equiv P(x, y) \equiv x^r y^s$$

and

$$Q \equiv Q(x, y)$$

for in that case the result is a terminating series due to the differentiators $p_1$, $p_2$.

(21) *An Illustrative Example:*

$$f(d_1, d_2) \equiv \frac{1}{d_1 + d_2}, \qquad P \cdot Q \equiv (x^2y^2) \cdot \log\,(x + y)$$

$$\begin{array}{lll}
p_1x^2y^2 = 2xy^2, & p_2x^2y^2 = 2x^2y, & p_1p_2x^2y^2 = 4xy \\
p_1^2x^2y^2 = 2y^2, & p_2^2x^2y^2 = 2x^2, & p_1^2p_2x^2y^2 = 4y \\
p_1^3x^2y^2 = 0, & p_2^3x^2y^2 = 0, & p_1p_2^2x^2y^2 = 4x \\
\cdots\cdots & \cdots\cdots & p_1^2p_2^2x^2y^2 = 4 \\
 & & \cdots\cdots
\end{array}$$

$$\begin{aligned}
\frac{\partial}{\partial q_1}\frac{1}{q_1 + q_2} &= \frac{-1}{(q_1 + q_2)^2} = \frac{\partial}{\partial q_2}\frac{1}{q_1 + q_2} \\
\frac{\partial^2}{\partial q_1^2}\frac{1}{q_1 + q_2} &= \frac{2}{(q_1 + q_2)^3} = \frac{\partial^2}{\partial q_2^2}\frac{1}{q_1 + q_2} \\
\frac{\partial^3}{\partial q_1^3}\frac{1}{q_1 + q_2} &= \frac{-6}{(q_1 + q_2)^4} = \frac{\partial^3}{\partial q_2^3}\frac{1}{q_1 + q_2} \\
\cdots & \cdots\cdots
\end{aligned}$$

The operations on $\log\,(x + y)$ can be simplified if we take

$$x + y = t, \qquad \text{then} \qquad q_1 + q_2 = q = \frac{d}{dt}$$

and

$$\frac{1}{(q_1 + q_2)^k}\log\,(x + y) = \frac{1}{q^k}\log t = \frac{1}{q^{k+1}}\frac{1}{t}$$

Then successively

$$\begin{aligned}
\frac{1}{q^2}\frac{1}{t} &= \frac{1}{2}(t \log t - t) \\
\frac{1}{q^3}\frac{1}{t} &= \frac{1}{4}\left[t^2\left(\frac{1}{2}\log t - \frac{1}{4}\right) - \frac{1}{2}t^2\right]
\end{aligned}$$

etc.*

* See Peirce, Table of Integrals, No. 426.

Now, by our theorem,

$$\frac{1}{d_1 + d_2} \equiv \frac{1}{q_1 + q_2} - (p_1 + p_2)\frac{1}{(q_1 + q_2)^2}$$
$$+ (p_1^2 + 2p_1p_2 + p_2^2)\frac{1}{(q_1 + q_2)^3}$$
$$- (p_1^3 + 3p_1^2p_2 + 3p_1p_2^2 + p_2^3)\frac{1}{(q_1 + q_2)^4} +$$

We shall have then

$$\frac{1}{+}\ (x^2y^2)\log(x + y) = [x^2y^2] \cdot \left[\frac{1}{q^2}\frac{1}{t}\right]$$
$$- [(p_1 + p_2)x^2y^2] \cdot \left[\frac{1}{q^3}\frac{1}{t}\right] + [(p_1^2 + 2p_1p_2 + p_2^2)x^2y^2] \cdot \left[\frac{1}{q^4}\frac{1}{t}\right]$$
$$- [(3p_1^2p_2 + 3p_1p_2^2)x^2y^2] \cdot \left[\frac{1}{q^5}\frac{1}{t}\right] + [6p_1^2p_2^2x^2y^2] \cdot \left[\frac{1}{q^6}\frac{1}{t}\right]$$
$$= x^2y^2 \cdot \tfrac{1}{2}[(x + y)\log(x + y) - (x + y)]$$
$$- (2xy^2 + 2x^2y)\tfrac{1}{4}[\tfrac{1}{2}(x + y)^2\log(x + y) - \tfrac{3}{4}(x + y)^2]$$
$$+ (2y^2 + 8xy + 2x^2)\tfrac{1}{8}[\tfrac{1}{6}(x + y)^3\log(x + y) - \tfrac{11}{18}(x + y)^3]$$
$$- (12y + 12x)\tfrac{1}{16}[\tfrac{1}{24}(x + y)^4\log(x + y) - \tfrac{47}{288}(x + y)^4]$$
$$+ (24)\tfrac{1}{32}[\tfrac{1}{120}(x + y)^5\log(x + y) - \tfrac{247}{7200}(x + y)^5]$$

all other terms being zero.

(22) For the student, $\frac{1}{d_1 - d_2}x^3y\sin(2x - y)$

## §30. Elementary Interpretation.

(1) We have already defined the partial linear operators $d_1$, $d_2$. It remains to interpret the elementary algebraic forms which appear in practical applications. The direct operators need no interpretation, but the inverses do. The latter may be classified as either homogeneous or nonhomogeneous.

*A. Homogeneous Forms.* (2) Every homogeneous function of $d_1$, $d_2$ can be factored into the following types:

(a) $d_1 - md_2$

(b) $(d_1 - md_2)^p$

(c) $(d_1 - md_2)^2 + \beta^2d_2^2$

(d) $[(d_1 - md_2)^2 + \beta^2d_2^2]^p$

and since inverses can be separated into partial fractions of the forms of the inverses of the preceding, we have but to interpret their inverses.

(3) (*a*) $\dfrac{1}{d_1 - md_2}$ can be simplified at once by Theorem I (§29), as follows:

$$\frac{1}{d_1 - md_2} \equiv \frac{1}{d_1 - md_2} e^{mxd_2} e^{-mxd_2}, \qquad 1 \equiv e^{mxd_2}e^{-mxd_2}$$

$$\equiv e^{mxd_2}\frac{1}{d_1}e^{-mxd_2} \qquad \text{[Th. I]}$$

With a change of lettering, this is

$$\equiv e^{mxd_2}\frac{1}{\mu_1}e^{-mud_2}$$

$$\equiv \frac{1}{\mu_1}e^{m(x-u)d_2}$$

$$\equiv \int^x e^{m(x-u)d_2}()\,\partial u$$

Here $\mu_1^{-1}$ stands for the sign of integration with respect to $u$ with a substitution of $x$ for $u$ after integration.

(4) (*b*) $$\frac{1}{(d_1 - md_2)^p} \equiv \frac{1}{(d_1 - md_2)^p}e^{mxd_2}e^{-mxd_2}, \quad 1 \equiv e^{mxd_2}e^{-mxd_2}$$

$$\equiv e^{mxd_2}\frac{1}{d_1{}^p}e^{-mxd_2} \qquad \text{[Th. I]}$$

$$\equiv e^{mxd_2}\frac{1}{\mu_1{}^p}e^{-mud_2}, \qquad \text{[relettering]}$$

$$\equiv \frac{1}{\mu_1{}^p}e^{m(x-u)d_2}$$

$$\equiv \int^x\int^u \cdots \int^u e^{m(x-u)d_2}()\,\partial u^p$$

Here the multiple integral can be simplified into a single integral in the same way that we simplified the ordinary type [see II §6 (7)].

$$\equiv \int^x \frac{(x-u)^{p-1}}{(p-1)!}e^{m(x-u)d_2}()\,\partial u$$

$$\equiv \frac{1}{\mu_1}\frac{(x-u)^{p-1}}{(p-1)!}e^{m(x-u)d_2}$$

(5) (*c*) $$\frac{1}{(d_1 - md_2)^2 + \beta^2 d_2{}^2} \equiv e^{mxd_2}\frac{1}{d_1{}^2 + \beta^2 d_2{}^2}e^{-mxd_2}$$

Here we must interpret the form $\dfrac{1}{d_1^2 + \beta^2 d_2^2}$. By means of partial fractions, we have

$$\begin{aligned}
\frac{1}{d_1^2 + \beta^2 d_2^2} &\equiv \frac{1}{(d_1 + i\beta d_2)(d_1 - i\beta d_2)}, \qquad i \equiv \sqrt{-1} \\
&\equiv \frac{1}{2i\beta d_2}\left[\frac{1}{d_1 - i\beta d_2} - \frac{1}{d_1 + i\beta d_2}\right] \\
&\equiv \frac{1}{2i\beta d_2}\left[e^{i\beta x d_2}\frac{1}{d_1}e^{-i\beta x d_2} - e^{-i\beta x d_2}\frac{1}{d_1}e^{i\beta x d_2}\right] \\
&\equiv \frac{1}{2i\beta d_2}\cdot\left[\frac{1}{\mu_1}e^{i\beta(x-u)d_2} - \frac{1}{\mu_1}e^{-i\beta(x-u)d_2}\right] \\
&\equiv \frac{1}{\beta d_2}\cdot\frac{1}{\mu_1}\left[\frac{1}{2i}(e^{i\beta(x-u)d_2} - e^{-i\beta(x-u)d_2})\right] \\
&\equiv \frac{1}{\beta}\cdot\frac{1}{d_2}\cdot\frac{1}{\mu_1}\sin\beta(x-u)d_2
\end{aligned}$$

We can now go back to the original form

$$\begin{aligned}
\frac{1}{(d_1 - md_2)^2 + \beta^2 d_2^2} &\equiv e^{mxd_2}\cdot\frac{1}{\beta}\cdot\frac{1}{d_2}\cdot\frac{1}{\mu_1}\sin\beta(x-u)d_2\cdot e^{-mud_2} \\
&\equiv \frac{1}{\beta}\cdot\frac{1}{d_2}\cdot\frac{1}{\mu_1}e^{m(x-u)d_2}\cdot\sin\beta(x-u)d_2 \\
&\equiv \frac{1}{\beta}\int\partial y\int^x e^{m(x-u)d_2}\cdot\sin\beta(x-u)d_2()\partial u
\end{aligned}$$

In this, note that both $e^{m(x-u)d_2}$ and $\sin\beta(x-u)d_2$ are operators which can be expanded into power series in $d_2$ and therefore completely interpretable as linear operators in $d_2$. After these two operations are performed, there is an integration in $u$ and one in $y$.

(6)($d$) The last of the four forms can be quickly interpreted in the light of the foregoing, as follows:

$$\begin{aligned}
\frac{1}{[(d_1 - md_2)^2 + \beta^2 d_2^2]^p} &- e^{mxd_2}\frac{1}{[d_1^2 + \beta^2 d_2^2]^p}e^{-mxd_2} \\
&\equiv e^{mxd_2}\left[\frac{1}{\beta}\cdot\frac{1}{d_2}\cdot\frac{1}{\mu_1}\sin\beta(x-u)d_2\right]^p e^{-} \\
&\equiv \frac{1}{\beta^p}\left[\frac{1}{d_2}\cdot\frac{1}{\mu_1}\sin\beta(x-u)d_2\right]^p e^{m(x-u)d_2}
\end{aligned}$$

(7) *Examples:* Make up integral forms for the following:

(a)
$$\begin{aligned}
\frac{1}{d_1 - d_2} f(x, y) &\equiv e^{xd_2}\frac{1}{d_1}e^{-xd_2}f(x, y) \\
&\equiv e^{xd_2}\frac{1}{d_1}f(x, y - x) \\
&\equiv e^{xd_2}\frac{1}{\mu_1}f(u, y - u) \\
&\equiv \frac{1}{\mu_1}e^{xd_2}f(u, y - u) \\
&\equiv \frac{1}{\mu_1}f(u, y + x - u) \\
&\equiv \int^x f(u, y + x - u)\partial u
\end{aligned}$$

(b) $\dfrac{1}{d_1 - d_2}\sin(x + y)$

(c) $\dfrac{1}{d_1 - d_2}e^{x-y}$

(d) $\dfrac{1}{(d_1 - 2d_2)^2}f(x, y)$

(e) $\dfrac{1}{(d_1 - 2d_2)^2}(x + y)$

(f) $\dfrac{1}{d_1^2 + d_2^2}x^2y$

(g) $\dfrac{1}{d_1^2 - d_2^2}xy^2$

(h) $\dfrac{1}{(d_1^2 + d_2^2)^2}xy$

*B. Nonhomogeneous Forms.* (8) The nonhomogeneous functions can all be factored into nonhomogeneous linear or quadratic factors of the following types:—

(a) $d_1 - md_2 - a$.
(b) $(d_1 - md_2 - a)^p$.
(c) $(d_1 - \alpha d_2)^2 + \beta^2$.
(d) $[(d_1 - \alpha d_2)^2 + \beta^2]^p$.
(e) $(d_1 - \alpha)^2 + (d_2 - \beta)^2$.

(9)(a) [Th. I]
$$\begin{aligned}
\frac{1}{d_1 - md_2 - a} &\equiv e^{x(md_2+a)}\frac{1}{d_1}e^{-x(md_2+a)} \\
&\equiv \frac{1}{\mu_1}e^{(x-u)(md_2+a)} \\
&\int^x e^{(x-u)(md_2+a)}()\partial u
\end{aligned}$$

(10)(b)
$$\frac{1}{(d_1 - md_2 - a)^p} \equiv e^{x(md_2+a)}\frac{1}{d_1{}^p}e^{-x(md_2+a)}$$
$$\equiv \frac{1}{\mu_1{}^p}e^{(x-u)(md_2+a)}$$
$$\equiv \int^x \int^u \cdots \int^u e^{(x-u)(md_2+a)}()\partial u^p$$
$$\equiv \int^x \frac{(x-u)^{p-1}}{(p-1)!}e^{(x-u)(md_2+a)}()\partial u$$
$$\equiv \frac{1}{\mu_1}\frac{(x-u)^{p-1}}{(p-1)!}e^{(x-u)(md_2+a)}$$

(11)(c)
$$\frac{1}{(d_1 - \alpha d_2)^2 + \beta^2} \equiv e^{\alpha x d_2} \cdot \frac{1}{d_1{}^2 + \beta^2} \cdot e^{-\alpha x d_2}$$

Since $\frac{1}{d_1{}^2 + \beta^2}$ is an operator in only one variable, it is interpreted as in the similar form in $D$:

$$\frac{1}{d_1{}^2 + \beta^2} \equiv \frac{1}{\beta} \cdot \frac{1}{\mu_1} \sin \beta(x - u)$$
$$\equiv \frac{1}{\beta}\int^x \sin \beta(x - u)()\partial u$$

Then we shall have

$$\frac{1}{(d_1 - \alpha d_2)^2 + \beta^2} \equiv e^{\alpha x d_2}\frac{1}{\beta}\frac{1}{\mu_1} \sin \beta(x - u) \cdot e^{-\alpha u d_2}$$
$$\equiv \frac{1}{\beta} \cdot \frac{1}{\mu_1} \sin \beta(x - u) \cdot e^{\alpha(x-u)d_2}$$
$$\equiv \frac{1}{\beta}\int^x \sin \beta(x - u)e^{\alpha(x-u)d_2}()\partial u$$

(12)(d)

$$\frac{1}{[(d_1 - \alpha d_2)^2 + \beta^2]^p} \equiv e^{\alpha x d_2}\frac{1}{[d_1{}^2 + \beta^2]^p} \quad {}^{\alpha x d_2}$$
$$\equiv \frac{1}{[\mu_1{}^2 + \beta^2]^p} \cdot e^{\alpha(x-u)d_2}$$
$$\equiv \frac{1}{\beta^p}\left[\frac{1}{\mu_1} \sin \beta(x - u)\right]^p \cdot e^{\alpha(x-u)d_2}$$

(13)(*e*)

$$\frac{1}{(d_1-\alpha)^2+(d_2-\beta)^2} \equiv e^{\alpha x+\beta y}\frac{1}{d_1^2+d_2^2}e^{-(\alpha x+\beta y)}$$
$$\equiv e^{\alpha x+\beta y}\left[\frac{1}{d_2}\frac{1}{\mu_1}\sin\,(x-u)d_2\right]e^{-(\alpha u+\beta v)}$$
$$\equiv \frac{1}{\mu_2}\frac{1}{\mu_1}\sin\,(x-u)d_2\cdot e^{\alpha(x-u)+\beta(y-v)}$$
$$\int^y\partial v\int^x\partial u\cdot\sin\,(x-u)d_2\cdot e^{\alpha(x-u)+\beta(y-v)}\,()$$

(14) *Interpret*:

(*a*) $\dfrac{1}{\rho d_1^2 - Ed_2^2}$

(*b*) $\dfrac{}{d_1^3 + d_1^2d_2 - d_1d_2^2 - d_2^3}$

(*c*) $\dfrac{1}{d_1^2 + 2d_1d_2 + d_2^2}$

(*d*) $\dfrac{1}{d_1d_2}$

(*e*) $\dfrac{}{d_1d_2 - d_2^2}$

(*f*) $\dfrac{1}{d_1 + d_2 - 1}$

(*g*) $\dfrac{1}{d_1^2 -}$

(*h*) $\dfrac{}{d_1^2 + d_2^2 + 1}$

(*i*) $\dfrac{1}{d_1^2 - d_2^2 + d_1 + d_2}$

(*j*) $\dfrac{1}{d_1d_2(d_1 - 2d_2 - 3)}$

(*k*) $\dfrac{1}{d_1 - a^2d_2^2}$

(*l*) $\dfrac{1}{d_1^2 - d_1d_2 + d_2 - 1}$

## §31. Operations on Zero.

*A. Homogeneous.* (1) Let us refer back to Chap. VII [§28 (1–3)], the elementary definition of $d \equiv \dfrac{\partial}{\partial x}$ and its inverse. We said, in our study of the ordinary operator $D$ [II, §7 (1)], that in operations on zero the appendage was desired. In the partial inverses on zero, this is also true,but the appendages here take on quite different forms. Whereas, in the ordinary operations, they bring in arbitrary constants multiplying the powers of the independent variable, here *arbitrary functions* are brought in.

(2) Since

$$d_1^{-1}S = \int S\partial x + \phi(y)$$
$$= \int S\partial x + \int 0\partial x$$
$$= \int (S+0)\partial x = d_1^{-1}(S+0)$$

we have

$$d_1^{-1} \cdot 0 = \phi(y)$$

also, similarly,

$$d_2^{-1} \cdot 0 = f(x)$$

(3) Then

$$\begin{aligned} d_1^{-2} \cdot 0 &= d_1^{-1}[\phi_1(y) + 0] \\ &= x\phi_1(y) + \phi_0(y) \end{aligned}$$

Each iteration brings in a different arbitrary function. We shall thus have

$$d_1^{-n} \cdot 0 = \sum_{k=0}^{n-1} x^k \phi_k(y) .$$

and, similarly,

$$d_2^{-n} \cdot 0 = \sum_{k=0}^{-} y^k f_k(x)$$

(4) Since

$$\frac{1}{d_1^m d_2^n} \equiv \sum_{k=m}^{1} \frac{A_k}{d_1^k} + \sum_{k=n}^{1} \frac{B_k}{d_2^k}$$

we have

$$\begin{aligned} \frac{1}{d_1^m d_2^n} \cdot 0 &= \sum_{k=m}^{1} \frac{1}{d_1^k} \cdot 0 + \sum^{1} \frac{1}{d_2^k} \cdot 0 \\ &\sum_{k=m-1}^{0} x^k \phi_k(y) + \sum_{k=n-1}^{\smile} y^k f_k(x) \end{aligned}$$

(5)

$$\begin{aligned} \frac{1}{d_1 - \alpha d_2} 0 &= e^{\alpha x d_2} \frac{1}{d_1} e^{-\alpha x d_2} 0 \\ &= e^{\alpha x d_2} \frac{1}{d_1} 0 = e^{\alpha x d_2} \phi(y) \\ &= \phi(y + \alpha x) \end{aligned}$$

Also,

$$
\begin{aligned}
\frac{1}{(d_1 - \alpha d_2)^p}0 &= e^{\alpha x d_2}\frac{1}{d_1{}^p}e^{-\alpha x d}\,0 \\
&= e^{\alpha x d_2}\frac{1}{d_1{}^p}0 \\
&= e^{\alpha x d_2}\sum_{k=p-1} x^k\phi_k(y) \\
&= \sum_{k=p-1} x^k\phi_k(y + \alpha x)
\end{aligned}
$$

(6) $$\prod_{k=1}\frac{1}{d_1 - \alpha_k d_2}\cdot 0, \qquad \text{by partial fractions}$$

$$
\begin{aligned}
&\equiv \sum_{k=1}^{n} N_k\frac{1}{d_1 - \alpha_k d_2}\cdot 0 \\
&= \sum_{k=1}^{n} N_k\phi_k(y + \alpha_k x), \qquad \text{by (5)}
\end{aligned}
$$

or, by absorbing the $N_k$ in the $\phi_k$,

$$= \sum_{k=1}^{n}\phi_k(y + \alpha_k x)$$

Here the $N_k$ are made up of constants and the operator $d_2$, the latter of which changes $\phi_k$, but the end result is simply a different function of $y + \alpha_k x$, so that $\phi_k$ is just as usable as any other, since it is arbitrary.

(7) $$\frac{1}{d_1{}^2 + \beta^2 d_2{}^2}\cdot 0$$

Since

$$
\begin{aligned}
\frac{}{d_1{}^2 + \beta^2 d_2{}^2} &= \frac{}{(d_1 + i\beta d_2)(d_1 - i\beta d_2)} \\
&\equiv \frac{1}{2i\beta d_2}\left[\frac{1}{d_1 - i\beta d_2} - \frac{1}{d_1 + i\beta d_2}\right]
\end{aligned}
$$

we shall have

$$\frac{1}{d_1^2 + \beta^2 d_2^2} \cdot 0 = \frac{1}{2i\beta d_2}\left[\frac{}{d_1 - i\beta d_2} \cdot 0 - \frac{}{d_1 + i\beta d_2} \cdot 0\right]$$
$$= \frac{1}{2i\beta d_2}\left[e^{i\beta x d_2} \cdot \frac{1}{d_1}0 - e^{-i\beta x d_2}\frac{1}{d_1}0\right]$$
$$= \frac{1}{2i\beta d_2}[e^{i\beta x d_2}\phi_1(y) + e^{-i\beta x d_2}\phi_2(y)]$$
$$= \frac{1}{2i\beta d_2}[\phi_1(y + i\beta x) + \phi_2(y - i\beta x)]$$

Now, since $2i\beta$ is a constant, and $d_2^{-1}$ is an integration with respect to $y$, we shall have two similar forms

$$= \Phi_1(y + i\beta x) + \Phi_2(y - i\beta x)$$

so that we could ignore the partial fraction coefficients and write

$$\frac{1}{d_1^2 + \beta^2 d_2^2} \cdot 0 = \Phi_1(y + i\beta x) + \Phi_2(y - i\beta x)$$

We may obtain another form for this operation, as follows:

$$\frac{1}{d_1^2 + \beta^2 d_2^2} \cdot 0 = e^{i\beta x d_2}\phi_1(y) + e^{-i\beta x d_2}\phi_2(y)$$
$$= [\cos \beta x d_2 + i \sin \beta x d_2]\phi_1(y) + [\cos \beta x d_2 - i \sin \beta x d_2]\phi_2(y)$$
$$= \cos \beta x d_2[\phi_1(y) + \phi_2(y)] + i \sin \beta x d_2[\phi_1(y) - \phi_2(y)]$$

and if

$$\phi_1(y) + \phi_2(y) = A(y)$$
$$i[\phi_1(y) - \phi_2(y)] = B(y)$$

we have

$$\frac{1}{d_1^2 + \beta^2 d_2^2} \cdot 0 = (\cos \beta x d_2)A(y) + (\sin \beta x d_2)B(y)$$
$$= \left[1 - \frac{\beta^2 x^2}{2!}d_2^2 + \frac{\beta^4 x^4}{4!}d_2^4 - + \cdots\right]A(y)$$
$$\left[\beta x d_2 - \frac{\beta^3 x^3}{3!}d_2^3 + \frac{\beta^5 x^5}{5!}d_2^5 - + \cdots\right]B(y)$$
$$= A(y) - \frac{\beta^2 x^2}{2!}A''(y) + \frac{\beta^4 x^4}{4!}A^{iv}(y) - + \cdots$$
$$+ \beta x B'(y) - \frac{\beta^3 x^3}{3!}B'''(y) + \frac{\beta^5 x^5}{5!}B^{(v)}(y) - + \cdots$$

This latter form should be interesting in the evaluation of the arbitrary functions from terminal conditions.

(8) $\dfrac{1}{(d_1^2 + \beta^2 d_2^2)^p} \cdot 0$. With (7) at hand, we can easily see how this form can be evaluated. Since

$$\frac{1}{(d_1^2 + \beta^2 d_2^2)^p} \equiv \sum_{k=1}^{p} \left[ \frac{N_{1k}}{(d_1 + i\beta d_2)^k} + \frac{N_{2k}}{(d_1 - i\beta d_2)^k} \right]$$

we have

$$\begin{aligned}
\frac{1}{(d_1^2 + \beta^2 d_2^2)^p} \cdot 0 &= \sum_{k=1}^{p} \left[ \frac{1}{(d_1 + i\beta d_2)^k} \cdot 0 + \frac{1}{(d_1 - i\beta d_2)^k} \cdot 0 \right] \\
&= \sum_{k=1}^{p} \left[ e^{-i\beta x d_2} \frac{1}{d_1^k} \cdot 0 + e^{i\beta x d_2} \frac{1}{d_1^k} \cdot 0 \right] \\
&= \sum_{k=1}^{p} \left[ e^{-i\beta x d_2} \sum_{h=1}^{k} x^h \phi_{hk}(y) + e^{i\beta x d_2} \sum_{h=1}^{k} x^h \psi_{hk}(y) \right]
\end{aligned}$$

Here, since $k$ runs from 1 to $p$, all forms in $h < k$ can be ignored, since they will be found repeated [Chap. II §7 (8)]; so that we have

$$= \sum_{k=0}^{p-1} [x^k \phi_{1k}(y - i\beta x) + x^k \phi_{2k}(y + i\beta x)]$$

or, if the trigonometric form is desired,

$$= (\cos \beta x d_2) \sum_{k=1}^{p} x^k A_k(y) + (\sin \beta x d_2) \sum_{k=1}^{p} x^k B_k(y)$$

(9) Now we can dispose quickly of the rest of the forms by the use of Theorem I, as follows:

$$\frac{1}{(d_1 - \alpha d_2)^2 + \beta^2 d_2^2} \cdot 0 = e^{\alpha x d_2} \frac{1}{d_1^2 + \beta^2 d_2^2} \cdot 0; \qquad \text{then see (7)}$$

$$\frac{1}{[(d_1 - \alpha d_2)^2 + \beta^2 d_2^2]^p} \cdot 0 = e^{\alpha x d_2} \frac{1}{(d_1^2 + \beta^2 d_2^2)^p} \cdot 0; \qquad \text{then see (8)}$$

(10) Owing to the fact that the integration or differentiation with respect to either independent variable does not change the

argument in the arbitrary functions, we can turn products of forms like $\frac{1}{(d_1 - \alpha d_2)^2 + \beta^2 d_2^2}$ into sums, as follows:

$$\prod \frac{1}{(d_1 - \alpha_k d_2)^2 + \beta_k^2 d_2^2} \cdot 0 = \sum_k \frac{1}{(d_1 - \alpha_k d_2)^2 + \beta_k^2 d_2^2} \cdot 0$$

$$= \sum_k e^{\alpha_k x d_2} \frac{1}{d_1^2 + \beta_k^2 d_2^2} \cdot 0; \qquad \text{then use (7)}$$

(11) For the same reason, products of quadratic inverses can be turned into sums.

$$\prod_k \frac{1}{[(d_1 - \alpha_k d_2)^2 + \beta_k^2 d_2^2]^{p_k}} \cdot 0$$

$$= \sum_k \left[ \sum_{j=1}^{p_k} \frac{1}{[(d_1 - \alpha_k d_2)^2 + \beta_k^2 d_2^2]^j} \cdot 0 \right.$$

$$= \sum_k \sum_{j=1}^{p_k} e^{\alpha_k x d_2} \frac{1}{(d_1^2 + \beta_k^2 d_2^2)^j} \cdot 0 \Bigg]; \qquad \text{then use (8)}$$

*B. Nonhomogeneous.* (12) It suffices merely to list the types of nonhomogeneous inverses in their operations on zero and to show their similarity of handling to the homogeneous forms.

(a) $(d_1 - md_2 - a)^{-1} \cdot 0 \equiv e^{x(a+md_2)} d_1^{-1} \cdot 0.$

(b) $(d_1 - md_2 - a)^{-p} \cdot 0 \equiv e^{x(a+md_2)} d_1^{-p} \cdot 0.$

(c) $[(d_1 - \alpha d_2)^2 + \beta^2]^{-1} \cdot 0 \equiv e^{\alpha x d_2}[d_1^2 + \beta^2]^{-1} \cdot 0.$

(d) $[(d_1 - \alpha d_2)^2 + \beta^2]^{-p} \cdot 0 \equiv e^{\alpha x d_2}[d_1^2 + \beta^2]^{-p} \cdot 0.$

(e) $[(d_1 - \alpha)^2 + (d_2 - \beta)^2]^{-1} \cdot 0 \equiv e^{\alpha x + \beta y}[d_1^2 + d_2^2]^{-1} \cdot 0.$

(f) $[(d_1 - \alpha)^2 + (d_2 - \beta)^2]^{-p} \cdot 0 \equiv e^{\alpha x + \beta y}[d_1^2 + d_2^2]^{-p} \cdot 0.$

In all these, the latter parts can be interpreted and evaluated from what has preceded. The student should obtain the final results in all cases.

(13) *Examples:*

(a) $\frac{1}{d_1^2 + d_1 d_2} \cdot 0$ (c) $\frac{1}{d_1^2 - a^2 d_2^2} \cdot 0$

(b) $\frac{1}{d_1^2 - d_2^2}\ 0$ (d) $\frac{1}{(d_1 + d_2)^2} \cdot 0$

(e) $\dfrac{1}{d_1^2 - 4d_1d_2 + 4d_2^2} \cdot 0$

(f) $\dfrac{1}{4d_1^2 + 12d_1d_2 + 9d_2^2} \cdot 0$

(g) $\dfrac{1}{d_1^2 + d_2^2}$

(h) $\dfrac{1}{d_1^3 + d_1d_2 - d_1d_2^2 - d_2^3} \cdot 0$

(i) $\dfrac{1}{d_1^3 - 3d_1^2d_2 + 2d_1d_2^2} \cdot 0$

(j) $\dfrac{1}{d_1^2 - d_1d_2 - 6d_2^2}$

(k) $\dfrac{1}{(d_1^2 + d_1)^2} \cdot 0$

(l) $\dfrac{1}{d_1 + d_2 - 1} \cdot 0$

(m) $\dfrac{1}{d_1 + d_2 - m}$

(n) $\dfrac{1}{d_1d_2 - 1} \cdot 0$

(o) $\dfrac{1}{d_1^2 - d_2^2 + d_1 + d_2}$

(p) $\dfrac{1}{(d_1 - d_2 - 1)(d_1 - 2d_2 - 1)} \cdot 0$

(q) $\dfrac{1}{d_1d_2(d_1 - 2d_2 - 3)} \cdot 0$

(r) $\dfrac{1}{d_1^2 - d_1d_2 + d_1} \cdot 0$

(s) $\dfrac{1}{(d_1 - 3d_2 - 2)^2} \cdot 0$

## §32. Operations on Unity.

*A. Homogeneous.* (1) The partial inverses $d_1^{-p}$ and $d_2^{-p}$ upon unity are interpreted and evaluated from the elementary definitions, *viz.*:

$$d_1^{-p} \cdot 1 = \frac{x^p}{p!}, \qquad d_2^{-p} \cdot 1 = \frac{y^p}{p!}$$

(2) When we come, however, to the more complicated forms, there are several methods of attack each of which produces a result that is valid.

(3) $(d_1 - \alpha d_2)^{-1}$. Since, by Theorem I,

$$(d_1 - \alpha d_2)^{-1} \equiv e^{\alpha x d_2} d_1^{-1} e^{-\alpha x d_2}$$

we have

$$\begin{aligned}(d_1 - \alpha d_2)^{-1} \cdot 1 &= e^{\alpha x d_2} d_1^{-1} e^{-\alpha x d_2} 1 \\ &= e^{\alpha x d_2} d_1^{-1} 1 = e^{\alpha x d_2} x = x\end{aligned}$$

Since

$$e^{-\alpha x d_2} 1 = (1 - \alpha x d_2 + \cdots)1 = 1$$

and

$$e^{\alpha x d_2} x = (1 + \alpha x d_2 + \cdots)x = x$$

or we may have

$$\begin{aligned}(d_1 - \alpha d_2)^{-1}1 &\equiv \frac{1}{-\alpha}\left(d_2 - \frac{1}{\alpha}d_1\right)^{-1}1\\ &= -\frac{1}{\alpha}\cdot e^{\frac{1}{\alpha}yd_1}\frac{1}{d_2}e^{-\frac{1}{\alpha}yd_1}1\\ &= -\frac{1}{\alpha}\cdot e^{\frac{1}{\alpha}yd_1}\frac{1}{d_2}1\\ &= -\frac{1}{\alpha}e^{\frac{1}{\alpha}yd_1}y = -\frac{y}{\alpha}\end{aligned}$$

we might even say that one-half the sum of these two is the proper result; for, since $d_1 - \alpha d_2$ is a homogeneous function of the operators, the result should be homogeneous in $x$, $y$. We should then have

$$(d_1 - \alpha d_2)^{-1}\cdot 1 = -\frac{1}{2\alpha}(y - \alpha x)$$

The validity of this result is shown by the direct operation $d_1 - \alpha d_2$ upon it.

(4) The same method may be applied to the operation

$$(d_1 - \alpha d_2)^{-p}\cdot 1.$$

$$\begin{aligned}(d_1 - \alpha d_2)^{-p}\cdot 1 &= e^{\alpha x d_2}d_1^{-p}e^{-\alpha x d_2}1\\ &= e^{\alpha x d_2}d_1^{-p}1 = e^{\alpha x d_2}\frac{x^p}{p!} = \frac{x^p}{p!}\end{aligned}$$

Or we may have

$$\begin{aligned}(d_1 - \alpha d_2)^{-p}1 &\equiv \frac{1}{(-\alpha)^p}\left(d_2 - \frac{1}{\alpha}d_1\right)^{-p}1\\ &= \frac{1}{(-\alpha)^p}e^{\frac{1}{\alpha}yd_1}d_2^{-p}e^{-\frac{1}{\alpha}yd_1}1\\ &= \frac{1}{(-\alpha)^p}e^{\frac{1}{\alpha}yd_1}\frac{1}{d_2^p}1 = \frac{1}{(-\alpha)^p}e^{\frac{1}{\alpha}yd_1}\frac{y^p}{p!}\\ &= \frac{1}{(-\alpha)^p}\frac{y^p}{p!}\end{aligned}$$

Then, also,

$$(d_1 - \alpha d_2)^{-p}\cdot 1 = \frac{1}{2(-\alpha)^p p!}[y^p + (-\alpha x)^p]$$

or even

$$\frac{1}{(-2\alpha)^p p!}(y - \alpha x)^p$$

as can be found true by the direct operation.

The two-term homogeneous form can be obtained very simply by using §27 (5) (I$c$), thus: Use separately $1 \equiv e^{0y}$ and $1 \equiv e^{0x}$.

$$(d_1 - \alpha d_2)^p \cdot 1 \equiv (d_1 - \alpha d_2)^{-p} e^{0y} \equiv e^{0y} d_1{}^{-p} 1 = \frac{x^p}{p!}$$

$$(d_1 - \alpha d_2)^{-p} 1 \equiv (d_1 - \alpha d_2)^{-p} e^{0x} \equiv e^{0x} \frac{1}{(-\alpha)^p} \cdot \frac{1}{d_2{}^p} 1 = \frac{1}{(-\alpha)^p} \cdot \frac{y^p}{p!}$$

Both could not have been used together, for

$$(d_1 - \alpha d_2)^{-p} 1 \equiv (d_1 - \alpha d_2)^{-p} e^{0x+0y} = \frac{1}{0} 1 = \infty$$

(5) Using the last method of (4) upon $\prod_k (d_1 - \alpha_k d_2)^{-1} \cdot 1$, we have

$$\prod_{k=1}^{m} (d_1 - \alpha_k d_2)^{-1} \cdot 1 \equiv \frac{1}{d_1{}^m} 1 = \frac{x^m}{m!}$$

or

$$\prod_{k=1}^{m} (d_1 - \alpha_k d_2)^{-1} \cdot 1 \equiv \prod_{k=1}^{m} \frac{1}{-\alpha_k d_2} 1 = \frac{1}{\prod_{k=1}^{m} (-\alpha_k)} \cdot \frac{y^m}{m!}$$

from which, also,

$$\prod_{k=1}^{m} (d_1 - \alpha_k d_2)^{-1} \cdot 1 = \frac{1}{2}\left| \frac{x^m}{m!} + \frac{1}{\prod_{k=1}^{m} (-\alpha_k)} \frac{y^m}{m!} \right.$$

This operation may also be written

$$\prod_{k=1}^{m} (d_1 - \alpha_k d_2)^{-1} 1 \equiv \sum_{k=1}^{m} (d_1 - \alpha_k d_2)^{-1} 1$$

$$\equiv \sum_{k=1}^{m} \frac{1}{-2\alpha_k} (y - \alpha_k x)$$

(6) For $\prod_{k=1}^{m}(d_1 - \alpha_k d_2)^{-p_k} \cdot 1$, the three forms will be

$$(a)\quad \frac{1}{\left(\sum_1^m p_k\right)!}x^{\sum_1^m p_k}$$

$$(b)\quad \frac{1}{\prod_{k=1}^{m}(-\alpha_k)^{p_k}} \cdot \frac{y^{\sum_1^m p_k}}{\left(\sum_1^m p_k\right)!}$$

$$(c)\quad \frac{1}{\left(\sum_1^m p_k\right)!}\left[x^{\sum_1^m p_k} + \frac{1}{\prod_{k=1}^{m}(-\alpha_k)^{p_k}}y^{\sum_1^m p_k}\right]$$

The student should verify these. If the forms $(y - \alpha_k x)^{p_k}$ are wanted, what would be the form of the constant multipliers?

(7) The student should derive all of the following:

$$(a)\quad (d_1^2 + \beta^2 d_2^2)^{-m} \cdot 1 = \frac{x^{2m}}{(2m)!} = \frac{y^{2m}}{\beta^{2m}(2m)!}$$
$$= \frac{1}{2\beta^{2m}(2m)!} \cdot [y^{2m} + (\beta x)^{2m}]$$

$$(b)\quad [(d_1 - \alpha d_2)^2 + \beta^2 d_2^2]^{-m} \cdot 1 = \frac{x^{2m}}{(2m)!} = \frac{(y + \alpha x)^{2m}}{\beta^{2m}(2m)!}$$
$$= \frac{1}{2\beta^{2m}(2m)!}[(y + \alpha x)^{2m} + (\beta x)^{2m}]$$
$$= \frac{1}{2(\alpha^2 + \beta^2)^m(2m)!}[y^{2m} + (\alpha^2 + \beta^2)^m x^{2m}]$$

$$(c)\quad \prod_{k=1}^{m}[(d_1 - \alpha_k d_2)^2 + \beta_k^2 d_2^2]^{-p_k} \cdot 1$$
$$= \frac{x^{2\Sigma p_k}}{(2\Sigma p_k)!} = \frac{y^{2\Sigma p_k}}{(2\Sigma p_k)!} \cdot \prod_{k=1}^{m}\frac{1}{(\alpha_k^2 + \beta_k^2)^{p_k}}$$
$$= \frac{1}{2\prod_{k=1}^{m}(\alpha_k^2 + \beta_k^2)^{p_k}(2\Sigma p_k)!}\left[y^{2\Sigma p_k} + x^{2\Sigma p_k}\prod_{k=1}^{m}(\alpha_k^2 + \beta_k^2)^{p_k}\right]$$

(8) Operate on unity with the group of operators in §31 (13) (*a*) to (*k*).

*B. Nonhomogeneous.* (9) Nonhomogeneous inverses are of two types: (*a*) those in which there is no constant term and (*b*) those in which there is a constant term. The first of these operating on unity is aided by the successive use of $e^{0y} \equiv 1$ and $e^{0x} \equiv 1$ and Theorem I and gives a nonhomogeneous form in $(x, y)$. The second uses at once $e^{0x+0y} \equiv 1$ and Theorem I*c*.

(10) *Examples of the First Type:*

$$(a)\quad \frac{1}{d_1^2 + d_2} \cdot 1 = \frac{1}{d_1^2} 1 = \frac{x^2}{2}$$

$$= \frac{1}{d_2} 1 = y$$

$$= \frac{1}{2}\left[y + \frac{x^2}{2}\right]$$

$$(b)\quad \frac{1}{d_1 - kd_2^2} \cdot 1 = \frac{1}{d_1} 1 = x$$

$$= \frac{1}{-kd_2^2} \cdot 1 = -\frac{1}{k} \cdot \frac{y^2}{2}$$

$$= \frac{1}{2}\left[x - \frac{y^2}{2k}\right]$$

$$(c)\quad \frac{1}{d_1^2 + d_1 d_2 + d_1} \cdot 1$$

$$(d)\quad \frac{1}{d_1^2 - 3d_1 d_2 + 2d_2} \cdot 1$$

$$(e)\quad \frac{1}{d_1^2 - d_2^2 + d_1 + d_2} \cdot 1$$

$$(f)\quad \frac{1}{d_1^3 - d_2^2} \cdot 1$$

$$(g)\quad \frac{1}{d_1 - a^4 d_2^4} \cdot 1$$

(11) *Examples of the Second Type:*

$$(a)\quad \frac{1}{d_1^3 - 3d_1 d_2 + d_1 + 1} \cdot 1 = 1$$

$$(b)\quad \frac{1}{d_1^2 + 2d_1 d_2 + d_2^2 + a^2} \cdot 1 = \frac{1}{a^2}$$

$$(c)\quad \frac{1}{(d_1 - d_2 - 1)(d_1 - d_2 - 2)} \cdot 1$$

$$(d)\quad \frac{1}{(d_1 - 3d_2 - 3)^2} \cdot 1$$

(e) $\dfrac{1}{(d_1 + d_2 - 3)^2 \, (d_1 + 2d_2 - 5)^3} \cdot 1$

(f) $\dfrac{1}{(d_1 - 2d_2)^2 + 3^2} \cdot 1$

(g) $\dfrac{1}{[(d_1 - d_2)^2 - 5][(d_1 - 2d_2)^2 + 3^2]^2} \cdot 1$

**§33. Operations on Particular Functions.**

(1) As in II §9, we shall here tabulate the particular functions on which inverses may operate and the theorems that should be used.

| $F^{-1}(d_1, d_2) \cdot f(x, y)$ where | Use |
|---|---|
| (2) $f(x, y) \equiv K$ | $K \equiv K \cdot 1$; then see §32 |
| (3) $f(x, y) \equiv e^{ax+by}$ | §29, I*b*, or I*c*; or see (7) below |
| (4) $f(x, y) \equiv x^m y^n$ | Direct integration; or expand the operator and differentiate; or use §29, IV |
| (5) $f(x, y) \equiv e^{ax+by}\phi(x, y)$ | §29, I*b*; then according to the form $\phi(x, y)$ |
| (6) $f(x, y) \equiv x^m y^n \phi(x, y)$ | §29, IV |
| (7) $f(x, y) \equiv \phi(ax + by)$ | §29, III |
| (8) powers of trigonometric or logarithmic functions alone | The functions should first be reduced to first-degree functions or expanded into powers of the independent variables; then use (2)–(7) according to the form of $f(x, y)$ |

(9) *Examples:*

(a) $\dfrac{1}{d_1 d_2} \cdot a = a \dfrac{1}{d_1 d_2} 1 = axy$

(b) $\dfrac{1}{d_1 d_2} xy = \dfrac{1}{d_1} x \cdot \dfrac{1}{d_2} y = \dfrac{x^2}{2} \cdot \dfrac{y^2}{2} = \dfrac{x^2 y^2}{}$

(c) $\dfrac{1}{d_1^2 - 3d_1 d_2 + 2d_2^2} e^{x+2y} = \dfrac{1}{1^2 - 3 \cdot 1 \cdot 2 + 2 \cdot 2^2} e^{x+2y}$

$$= \frac{1}{3} e^{x+2y}$$

$$(d)\quad \frac{1}{2d_1^2 - d_1d_2 - 3d_2^2}5e^{x-y} = \frac{1}{(d_1 + d_2)(2d_1 - 3d_2)}5e^{x-y}$$

$$\frac{1}{d_1 + d_2} \cdot \frac{1}{2 \cdot 1 - 3(-1)}5e^{x-y} = \frac{1}{d_1 + d_2}e^{x-y}$$

$$= e^{-xd_2}\frac{1}{d_1}e^{xd_2}e^{x-y} = e^{-xd_2}\frac{1}{d_1}e^{x-(y+x)}$$

$$= e^{-xd_2}\frac{1}{d_1}e^{-y} = e^{-xd_2}xe^{-y} = xe^{-(y-x)}$$

$$= xe^{x-y}$$

$$(e)\quad \frac{1}{d_1 - ad_2}\, e^{mx} \cos ny$$

$$(f)\quad \frac{1}{d_1^2 - d_2^2}(y + e^{x+y})$$

$$(g)\quad \frac{\ }{d_1^2 - a^2d_2^2}$$

$$(h)\quad \frac{1}{2d_1^2 - d_1d_2 - 3d_2^2} \sin (x + y)$$

$$(i)\quad \frac{1}{d_1^2 + d_1d_2 - 6d_2^2}\, y \cos x$$

$$(j)\quad \frac{1}{d_2^2} \sin xy$$

$$(k)\quad \frac{1}{(d_1 - d_2)^2} \sin (2x + 3y)$$

$$(l)\quad \frac{1}{4d_1^2 - 4d_1d_2 + d_2^2}\, 16 \log (x + 2y)$$

$$(m)\quad \frac{1}{d_1^2 + d_2^2} \cos nx \cdot \cos my$$

$$(n)\quad \frac{1}{d_1 + d_2 - m}$$

$$(o)\quad \frac{1}{d_1 - d_2^2} \cos (x - 3y)$$

$$(p)\quad \frac{1}{d_1^2 - d_1d_2 - 2d_1} \sin (3x + 4y)$$

$$(q)\quad \frac{1}{d_1^2 - d_1d_2 + d_2 - 1}[\cos (x + 2y) + e^y]$$

# CHAPTER VIII

## APPLICATIONS TO PARTIAL LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

### §34. Physical Problems.

(1) When we examine the subjects of heat flow, string or rod vibration, wave propagation, and potential theory—two-dimensional problems—we find two independent variables and one dependent variable. Let us exhibit some of the most interesting of the differential equations encountered.

(2) The equation $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$ covers a great variety of physical situations:

*a.* Motion of a stretched string, finite or infinite length [Mellor].

*b.* Small oscillations of air in narrow pipes [Mellor].

*c.* The harmonic equation with $a^2 = -1$ [Bateman].

*d.* D'Alembert's equation, $a^2 = \frac{E}{\rho}$, where $E$ is the modulus of elasticity, and $\rho$ is the density of the material.

*e.* Velocity potential of waves in a canal with vertical sides [Ramsey].

*f.* Waves on a sea of medium depth [Mellor].

(3) The harmonic equation:

*a.* Two-dimensional heat flow and logarithmic potential:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \quad 0 \qquad \text{[Byerly]}$$

*b.* Poisson's equation in a plane:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \quad -4\pi\rho \qquad \text{[Murray]}$$

(4) The equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$.

*a*. Temperature of ocean at depth $x$ [Carr]
*b*. One-dimensional heat flow [Byerly]
(5) Motion in an imperfect fluid:

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = mz \qquad \text{[Mellor]}$$

(6) Lateral vibrations of bars, neglecting the inertia of rotation:

$$\frac{\partial^2 y}{\partial t^2} = -a^4\frac{\partial^4 y}{\partial x^4} \qquad \text{[Rayleigh]}$$

(7) The operational method for solving the equations of this type is simple and powerful, and we shall proceed at once to develop it.

**§35. Lagrange's Method.**

(1) This section deals with Lagrange's method of solution of the linear partial differential equation of the first order, and its analogy with the operational method. For the most part, it is the embodiment of two viewpoints as given by the following references:

*a*. Analytic treatment:

GOURSAT, "Mathematical Analysis," II-II Differential Equations, §31, p. 74; §75, p. 250.

COHEN, "Differential Equations," XIII 1, p. 250, ed. 1933.

*b*. Geometric treatment:

GOURSAT, *loc. cit.*, §76, p. 218.

OSGOOD, "Advanced Calculus," §22, p. 260.

(2) *The Analytic Treatment.* Given the two allied equations

$$P(x, y, z)\frac{\partial z}{\partial x} + Q(x, y, z)\frac{\partial z}{\partial y} = R(x, y, z) \qquad (a)$$

and

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0 \qquad (b)$$

Now, if $u(x, y, z) =$ constant satisfies ($a$), it will be a solution of ($b$); for, with

$$u(x, y, z) = c$$

then by implicit function theory

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$
$$\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$

and

$$\frac{\partial z}{\partial x} = -\frac{\partial u/\partial x}{\partial u/\partial z} \qquad \frac{\partial z}{\partial y} = -\frac{\partial u/\partial y}{\partial u/\partial z} \tag{c}$$

Substitute $(c)$ in $(a)$:

$$P\left[-\frac{\partial u/\partial x}{\partial u/\partial z}\right] + Q\left[-\frac{\partial u/\partial y}{\partial u/\partial z}\right] = R$$

giving $(b)$. The converse is also true.

(3) Given the set of equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{d}$$

By multiplication of numerator and denominator of these fractions, respectively, by $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ and by composition, we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz}{P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z}}$$

But also,

$$\frac{du}{0}$$

from which

$$u = c \qquad \text{a solution of } (d)$$

Conversely if $u = c$ satisfies $(d)$, it will satisfy $(b)$; for that gives

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \tag{e}$$

(4) Now, suppose

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = K$$

from which

$$\begin{aligned} dx &= K \cdot P \\ dy &= K \cdot Q \\ dz &= K \cdot R \end{aligned}$$

which, substituted in (*e*), gives

$$\frac{\partial u}{\partial x}KP + \frac{\partial u}{\partial y}KQ + \frac{\partial u}{\partial z}KR = 0$$

or

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0$$

(5) Hence the problem of solving (*b*) is reduced to that of solving (*d*), since every solution of (*b*) is a solution of (*d*), and conversely.

(6) *The Geometric Treatment.* The equation

$$Pp + Qq = R, \qquad p \equiv \frac{\partial z}{\partial x}, \qquad q \equiv \frac{\partial z}{\partial y} \tag{a}$$

has a solution

$$\phi(x, y, z) = c$$

which can be solved for

$$z = f(x, y) \tag{f}$$

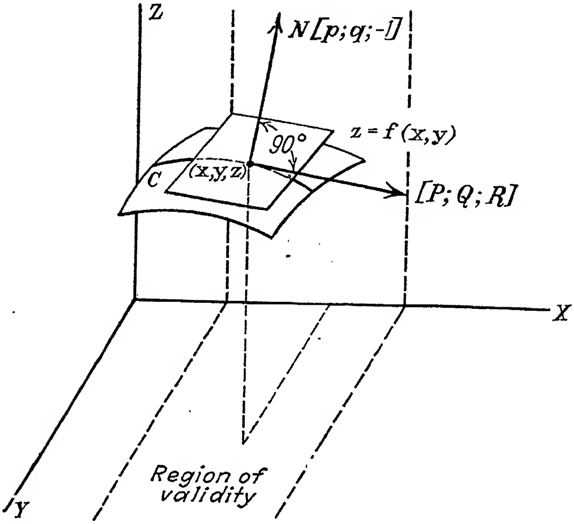
The direction cosines of the normal to (*f*) are

$$p;\ q;\ -1$$

Comparing Eq. (*a*) with the condition for perpendicularity for two lines, *viz.*,

$$\begin{aligned} Pp + Qq - R &= 0 \\ \cos\theta = ll' + mm' + nn' &= 0 \end{aligned}$$

we can easily see that $P; Q; R$ are the direction numbers of a direction in the tangent plane to the surface (*f*) and of the tangents of the curves represented by the Eqs. (*d*). Therefore solutions of (*d*) will give the surfaces (*f*), the desired solution of (*a*).




(7) Applying this method to the solution of the equation

$$\frac{\partial z}{\partial x} - m\frac{\partial z}{\partial y} = 0$$

we have

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{0}$$

and

$$dz = 0 \qquad z = c_1$$
$$-m\,dx = dy \qquad -mx + c_2 = y$$

whence

$$\phi(c_1, c_2) \equiv \phi[z, y + mx] = 0$$

or

$$z = f(y + mx)$$

(8) Applied to

$$\frac{\partial z}{\partial x} - m\frac{\partial z}{\partial y} = k$$

we have

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{k}$$

from which

$$c_1 = y + mx$$
$$c_2 = z - kx$$
$$c_3 = ky + mz$$

and

$$\phi(c_1, c_2) \equiv \phi[z - kx, y + mx]$$
$$z = kx + f(y + mx)$$

or

$$\phi(c_1, c_3) \equiv \phi[mz + ky, y + mx]$$
$$z = -\frac{k}{m}y + f(y + mx)$$

we might combine these and obtain

$$z = \frac{k}{2m}(y - mx) + f(y + mx)$$

(9) Now, note the operational solutions

$$(d_1 - md_2)z = 0$$
$$z = \frac{1}{d_1 - md_2} \cdot 0 = e^{mxd_2}f(y)$$
$$= f(y + mx)$$

and

$$(d_1 - md_2)z = k$$
$$z = \frac{1}{d_1 - md_2}[k + 0]$$
$$= \frac{k}{2}\left(\frac{y}{m} - x\right) + f(y + mx)$$

exactly the same forms as before.

(10) Take, now, the form

$$(d_1 - md_2)^2 z = 0$$

Substitute

$$(d_1 - md_2)z = u$$

giving

$$(d_1 - md_2)u = 0$$

from which

$$u = f(y + mx)$$

Now,

$$(d_1 - md_2)z = f(y + mx)$$

Use Lagrange's method on this:

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{f(c_3)} \qquad c_3 = y + mx$$

Integrals of this are

$$\begin{aligned} k_1 &= y + mx \\ k_2 &= z - xf(c_3) \end{aligned}$$

whence

$$\Phi(k_1, k_2) \equiv \Phi[z - xf(c_3), y + mx] = 0$$

*i.e.*,

$$z = xf(y + mx) + \phi(y + mx)$$

(11) Compare the direct operational solution

$$\begin{aligned} (d_1 - md_2)^2 z &= 0 \\ z &= \frac{1}{(d_1 - md_2)^2} \cdot 0 = e^{mxd_2}\frac{1}{d_1^2}0 \\ &= e^{mxd}\cdot[f_0(y) + xf_1(y)] \\ &= f_0(y + mx) + xf_1(y + mx) \end{aligned}$$

(12) Thus by an iterative use of Lagrange's method it is easy to show the parallelism between the results obtained by it and the operational method.

(13) Solve both by Lagrange's and by the operational method:

(*a*) $$\frac{\partial^2 u}{\partial t^2} \quad \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}$$

(*b*) $$\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial y} = 0$$

(*c*) $$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

(*d*) $$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

(*e*) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

**§36. The Single Equation.**

(1) Every linear partial differential equation with constant coefficients can be written in the form

$$F(d_1, d_2) \cdot z = f(x, y)$$

where

$$d_1 \equiv \frac{\partial}{\partial x}, \qquad d_2 \equiv \frac{\sigma}{\partial y}$$

and

$$F(d_1, d_2) \equiv \sum_{j,k} a_{jk} d_1{}^j d_2{}^k$$

the $a_{jk}$ being constants, real or complex, and the form homogeneous or nonhomogeneous in $d_1$, $d_2$. The largest value of $j + k$ is the order of the equation, and $j + k$ may tend to infinity. $F(d_1, d_2)$ may therefore be either a finite or infinite series, convergent or not. The $f(x, y)$ may be either a finite form or an infinite series.

(2) The operational solution requires the addition of zero to $f(x, y)$, thus:

$$F(d_1, d_2) \cdot z = f(x, y) + 0$$

Then the complete solution (general) will be obtained as in the ordinary equations.

$$z = \frac{1}{F(d_1, d_2)} \cdot f(x, y) + \frac{1}{F(d_1, d_2)} \cdot 0$$

The first form on the right is the *particular integral;* and the second, the *complementary function.*

(3) *The Complementary Function.* Full instructions for obtaining this are given in VII, §31.

(4) Where $F(d_1, d_2) = 0$ cannot be easily factored by algebraic methods, the method of solution by infinite series will be easier.

(5) *The Particular Integral.* For this, also, full instruction has been given in VII, §32.

(6) The student should be able to solve the following equations, many of which, though not specifically labeled, are equations arising in mathematical physics.

1. $(d_1 - \alpha d_2)z = 0$
2. $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{E}{\rho}\dfrac{\partial^2 u}{\partial x^2}$ [Mellor]
3. $d_1 d_2 z = 0$
4. $(d_1 d_2 - d_2)z = 0$
5. $(d_1^2 + d_1 d_2)z = 0$
6. $(d_1^2 - d_2^2)z = 0$
7. $\dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial x^2}$ [D'Alembert]
8. $(d_1^2 + 2d_1 d_2 + d_2^2)z = 0$
9. $(d_1^2 - 2d_1 d_2 + d_2^2)z = 0$
10. $(d_1^2 - 4d_1 d_2 + 4d_2^2)z = 0$
11. $(d_1^2 + 5d_1 d_2 + 2d_2^2)z = 0$
12. $(d_1^3 + d_1^2 d_2 - d_1 d_2^2 - d_2^3)z = 0$

13. $(d_1{}^3 - 3d_1{}^2d_2 + 2d_1d_2{}^2)z = 0$
14. $(d_1{}^3 - 6d_1{}^2d_2 + 11d_1d_2{}^2 - 6d_2{}^3)z = 0$
15. $(d_1{}^3 - 7d_1{}^2d_2 + 10d_1d_2{}^2)z = 0$
16. $(2d_1{}^4 - 3d_1{}^2d_2 + d_2{}^2)z = 0$
17. $(d_1{}^4 - 2d_1{}^2d_2{}^2 + d_2{}^4)z = 0$
18. $(d_1 - ad_2)z = e^{mx}\cos ny$
19. $(d_2 - d_1)z = y - x$
20. $(d_1{}^2 - d_2{}^2)z = \tan^3 x \tan y - \tan x \tan^3 y$
21. $\dfrac{\partial^2 y}{\partial x^2} - 4\dfrac{\partial^2 y}{\partial t^2} \quad \dfrac{4x}{t^2} \quad \dfrac{t}{x^2}$
22. $\dfrac{\partial^2 u}{\partial x^2} - a^2\dfrac{\partial^2 u}{\partial y^2} = x$
23. $(d_1{}^2 - a^2d_2{}^2)u = \phi(x, y)$
24. $d_1d_2z = 1$
25. $d_1d_2z = xy$
26. $d_1d_2z = \dfrac{x}{y} + a$
27. $d_1d_2z = 2x + 2y$
28. $(d_1{}^2 + 3d_1d_2 + 2d_2{}^2)z = x + y$
29. $(d_1{}^2 - d_1d_2 - 2d_2{}^2)z = x - y$
30. $(d_1{}^2 - 3d_1d_2 + 2d_2{}^2)z = e^{x+2y} + e^{x+y}$
31. $(d_1{}^2 - d_1d_2 - 6d_2{}^2)z = xy$
32. $(d_1{}^2 + d_1d_2 - 6d_2{}^2)z = y\cos x$
33. $d_2{}^2z = \sin xy$
34. $(d_1{}^2 - 2d_1d_2 + d_2{}^2)z = \sin(2x + 3y)$
35. $(4d_1{}^2 - 4d_1d_2 + d_2{}^2)z = 16\log(x + 2y)$
36. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = -4\pi\rho$
37. $(d_1{}^3 - 4d_1{}^2d_2 + 4d_1d_2{}^2)z = 4\sin(2x + y)$
38. $(d_1{}^2 + d_1)^2z = xy$
39. $(d_1{}^2 + d_2{}^2)z = \cos nx \cdot \cos ny$
40. $(d_1{}^2 + d_2{}^2)V = 12(x + y)$
41. $[a(d_1 + d_2) - 1]z = 0$
42. $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = mz$
43. $(d_1{}^2 - d_2)V = 0$
44. $\dfrac{\partial T}{\partial t} = K^2\dfrac{\partial^2 T}{\partial z^2}$
45. $(d_1{}^2 + d_1d_2 + d_1)z = 0$
46. $(d_1{}^2 - 3d_1d_2 + 2d_2)z = 0$
47. $(d_1{}^2 + d_2{}^2 + 1)V = 0$
48. $(d_1{}^2 + d_2{}^2 - n^2)V = 0$
49. $(d_1{}^2 - d_2{}^2 + d_1 + d_2)z = 0$
50. $(d_1{}^2 - d_2{}^2 + d_1 - d_2)z = 0$
51. $(d_1{}^2 - a^2d_2{}^2 + 2abd_1 + 2a^2bd_2)z = 0$
52. $(2d_1{}^2 - d_1d_2 - d_2{}^2 + 6d_1 + 3d_2)z = 0$
53. $(d_1{}^2 + 2d_1d_2 + d_2{}^2 + 2d_1 + 2d_2 + 1)z = 0$

54. $d_1d_2(d_1 - 2d_2 + 3)z = 0$

55. $(d_1^3 - d_2^2)z = 0$

56. $\dfrac{\partial u}{\partial t} = a^4\dfrac{\partial^4 u}{\partial x^4}$

57. $(d_1 + d_1^2)z = xy$

58. $(d_1 - d_2^2)z = \cos(x - 3y)$
59. $(d_1^2 - d_1d_2 - 2d_1)z = \sin(3x + 4y)$
60. $(d_1^2 - d_1d_2 + d_1)z = 1$
61. $(d_1d_2 + d_1 - d_2)z = z + xy$
62. $(d_1^3 - 3d_1d_2 + d_1 + 1)z = e^{2x+3y}$
63. $(d_1^2 - d_1d_2 + d_2 - 1)z = \cos(x + 2y) + e^y$
64. $(d_1^2 - d_2^2 + 2d_1 + 1)z = e^{-x}$
65. $(d_1^2 - d_2^2 + d_1 + 3d_2 - 2)z = e^{x-y} - x^2y$
66. $(d_1^2 + 2d_1d_2 + d_2^2 + a^2)z = \cos(mx + ny)$
67. $(d_1 - d_2 - 1)(d_1 - d_2 - 2)z = e^{2x-y}$
68. $(d_1 - 3d_2 - 2)^2z = 2e^{2x}\tan(y + 3x)$

**§37. Systems of Equations.**

(1) Apart from increased difficulties in evaluating arbitrary functions of two variables in the complementary functions, there is no reason why systems of partial differential equations with constant coefficients should not have the same procedures applied to their solution as to systems of ordinary equations of the same type, as discussed in Chap. VI.

(2) We shall, therefore, state the premises and the theorems as applied to partial systems and give a set of problems whose solutions can easily be obtained by them. Note the changes in the forms, substituting arbitrary functions for constants, and use, of course, the theorems of Chap. VII in the operational details.

(3) *The Complementary Functions.*
Given

$$\sum_{j=1}^{\cdots} F_{ij}(d_1, d_2)z_j = X_i(x, y), \qquad i = 1, 2, \cdots, m \qquad (a)$$

where $F_{ij}(d_1, d_2)$ is of form $\sum_{h,k} a_{hk}d_1^hd_2^k$: homogeneous or non-homogeneous in the operators; and $a_{hk}$ constants, real or complex. *First case*, homogeneous; *i.e.*, $X_i = 0$.

(4) **Theorem I. Each $z_j$ and the general complementary function satisfy the single differential equation**

$$F_{ij}(d_1, d_2) \mid \cdot V = 0 \tag{b}$$

**where the $\mid F_{ij}(d_1, d_2)$ is the determinant of the system (a) and is not identically zero in $d_1$ or $d_2$.**
$V$ is the *general complementary function* of the system and depends upon the linear factors in $d_1$ and $d_2$ of the *auxiliary equation*

$$F_{ij}(d_1, d_2) \mid = 0 \tag{c}$$

The *arbitrary functions* appearing in $V$ number $m$, the degree of the equation in either $d_1$ or $d_2$. The $\mid F_{ij} \mid$ is called the *characteristic determinant.*

(5) **Theorem II. The $z_j$ are proportional to the cofactors of the operator coefficients of any ith row in the characteristic determinant, the constant function of proportionality being the general complementary function V; i.e.,**

$$z_j = f_{ij}(d_1, d_2) \cdot V, \qquad j = 1, 2, \quad n \quad \text{for any } i \tag{d}$$

(6) *Case* i: *All Linear Factors Distinct.* Every root

$$d_1 = \alpha_k d_2 + \beta_k$$

gives the characteristic determinant a factor $d_1 - \alpha_k d_2 - \beta_k$ which is not a common factor of all the first minors. Thus not all the $f_{ij}$ in ($d$) can be made zero when they operate on $V$. We shall then have present after the operation all the arbitrary functions in $V$, and the $z_j$ are then the complementary functions of the general solution.

(7) *Case* ii: *Multiple Roots.* Every root $d_1 = \alpha_k d_2 + \beta_k$ of multiplicity $s + 1$ gives the characteristic determinant a factor $(d_1 - \alpha_k d_2 - \beta_k)^{s+1}$. When $(d_1 - \alpha_k d_2 - \beta_k)$ does not appear in all the $f_{ij}$, then no arbitrary functions are lost in ($d$). It can appear in all $f_{ij}$ to degree $s$ at most. Let $s$, however, represent the degree to which it does appear. Then we shall operate on ($d$) by $(d_1 - \alpha_k d_2 - \beta_k)^s$ and obtain a reduced set of proportions from which by further direct and inverse operations we can build up solutions containing the proper number of arbitrary functions; thus,

Using $(d)$,

$$\frac{z_j}{f_{ij}(d_1,\ d_2)} = V \qquad \begin{matrix} j = 1, 2, \quad \cdot\ , m \\ \text{for any } i \end{matrix} \tag{e}$$

and disclosing the common factor in the denominators,

$$\frac{z_j}{(d_1 - \alpha_k d_2 - \beta_k)^s g_{ij}(d_1,\ d_2)} = V \tag{f}$$

operate through by $(d_1 - \alpha_k d_2 - \beta_k)^s$, together with the use of $(d_1 - \alpha_k d_2 - \beta_k)^s (d_1 - \alpha_k d_2 - \beta_k)^{-s} \equiv 1$; we obtain

$$\frac{\quad}{g_{ij}(d_1,\ d_2)} = (d_1 - \alpha_k d_2 - \beta_k)^s \cdot V \tag{g}$$

Operate now by $(d_1 - \alpha_k d_2 - \beta_k)^p$, and use the substitution

$$(d_1 - \alpha_k d_2 - \beta_k)^p z_j = w_j, \qquad 1 \leqq p \leqq s \tag{h}$$

obtaining

$$\frac{w_j}{g_{ij}(d_1,\ d_2)} = (d_1 - \alpha_k d_2 - \beta_k)^{s+p} V = W \tag{i}$$

From this we have

$$w_j = g_{ij}(d_1,\ d_2) \cdot W \qquad \begin{matrix} j = 1, 2, \ \cdot \quad m \\ \text{for any } i \end{matrix} \tag{j}$$

Then by $(h)$ inversely,

$$z_j = \frac{\quad}{(d_1 - \alpha_k d_2 - \beta_k)^p}[w_j + 0] \tag{k}$$

(8) Here, the inverse operations upon $w_j$ will not lose any of the functions already obtained, and those upon zero will give for each $z_j$ $p$ additional ones. But we integrate only $k$ of the $z_j$, adding $kp$ arbitrary functions, $k$ being determined by the equation

$$n - (s + p) + kp = n$$

or

$$s + p = kp, \qquad [s = s \cdot k] \tag{l}$$

$k$ is an integer, and the equation $(l)$ sometimes gives a choice of $k$ and $p$.

(9) The other $m - k$ of the $z_j$ are to be obtained by using linear combinations of the $k$ ones integrated above. This is done by algebraic composition from the proportion ($g$):

$$\frac{z_j}{g_{ij}} = \frac{z_h}{g_{ih}}, \qquad \begin{array}{l} j, h = 1, 2, \cdots, m \\ j \neq h \\ \text{for any } i \end{array} \tag{m}$$

From this we have

$$\frac{z_j}{g_{ij}} = \frac{\Sigma z_h}{\Sigma g_{ih}}$$

or

$$z_j = \frac{g_{ij}}{\Sigma g_{ih}} \sum z_h \qquad \begin{array}{l} h = 1, 2, \cdots, k \\ j = k + 1, \cdots, m \\ j \neq h, \text{ for any } i \end{array} \tag{n}$$

The final result will be all the complementary functions, with the proper number $n$ of arbitrary functions demanded by the general solution.

(10) **Theorem III. Particular values of the unknowns z can be found for the equations (a) by the use of Cramer's rule; i.e.,**

$$z_j = | F_{ij} |^{-1} \cdot K_j \qquad j = 1, 2, \cdots, m \tag{o}$$

**where $K_j$ is the characteristic determinant $| F_{ij} |$ with the column of $X_i$ in place of its jth column.**

(11) *Illustrative Examples:*

(a) $$\begin{aligned} d_1 z_1 + d_2 z_2 &= k \\ -d_2 z_1 + d_1 z_2 &= 0 \end{aligned} \qquad \left\| \begin{array}{cc|c} d_1 & d_2 & k \\ -d_2 & d_1 & 0 \end{array} \right\|$$

$$\Delta \equiv \begin{vmatrix} d_1 & d_2 \\ -d_2 & d_1 \end{vmatrix} \equiv | F_{ij} | \equiv d_1^2 + d_2^2$$

$$\Delta \cdot V \equiv 0, \qquad V = \frac{1}{d_1^2 + d_2^2} \cdot 0 = f_1(y + ix) + f_2(y - ix)$$

$$\frac{z_1}{d_1} = \frac{z_2}{d_2} = V$$

$$z_1 = d_1 V = i[f_1'(y + ix) - f_2'(y - ix)]$$
$$z_2 = d_2 V = f_1'(y + ix) + f_2'(y - ix)$$

the complementary functions

$$\Delta\bar{z}_1 = \begin{vmatrix} k & d_2 \\ 0 & d_1 \end{vmatrix} = d_1 k$$

$$\bar{z}_1 = \frac{d_1}{d_1^2 + d_2^2}k = k\frac{d_1}{d_1^2 + d_2^2}1 = \frac{kx}{2}$$

$$\Delta\bar{z}_2 = \begin{vmatrix} d_1 & k \\ -d_2 & 0 \end{vmatrix} = d_2 k$$

$$\bar{z}_2 = \frac{d_2}{d_1^2 + d_2^2}k = k\frac{d_2}{d_1^2 + d_2^2}\cdot 1 = \frac{ky}{2}$$

Thus,

$$z_1 = if_1'(y + ix) - if_2'(y - ix) + \frac{kx}{2}$$

$$z_2 = f_1'(y + ix) + f_2'(y - ix) + \frac{ky}{2}$$

(b) $\left\| \begin{array}{cc|c} d_2 & d_2 & k \\ -d_1d_2 & d_1d_2 & 0 \end{array} \right\|$

$$\Delta \equiv \begin{vmatrix} d_2 & d_2 \\ -d_1d_2 & d_1d_2 \end{vmatrix} = d_1d_2^2\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2d_1d_2^2$$

$$\Delta \cdot V = 0, \qquad V = \frac{1}{\Delta}0 = \frac{1}{d_1d_2^2}0 = f_1(y) + f_2(x) + yf_3(x)$$

By cofactors of row 2,

$$\frac{z_1}{-d_2} = \frac{z_2}{d_2} = V$$

$$\frac{w_1}{-1} = \frac{w_2}{1} = d_2^2V = \frac{1}{d_1}0 = W = f_1(y)$$

$$w_1 = -f_1(y), \qquad z_1 = \frac{1}{d_2}[-f_1(y) + 0] = -F_1(y) + F_2(x)$$

$$w_2 = f_1(y), \qquad z_2 = \frac{1}{d_2}[f_1(y) + 0] = F_1(y) + F_3(x)$$

$$\Delta \cdot \bar{z}_1 = \begin{vmatrix} k & d_2 \\ 0 & d_1d_2 \end{vmatrix} = d_1d_2k$$

$$\bar{z}_1 = \frac{d_1d_2}{2d_1d_2^2}k = \frac{k}{2}\cdot\frac{1}{d_2}1 = \frac{ky}{2}$$

$$\Delta \cdot \bar{z}_2 = \begin{vmatrix} d_2 & k \\ -d_1d_2 & 0 \end{vmatrix} = \frac{d_1d_2k}{2d_1d_2^2} = \frac{ky}{2}$$

Thus,

$$z_1 = -F_1(y) + F_2(x) + \frac{ky}{2}$$

$$z_2 = \quad F_1(y) + F_3(x) + ky$$

(c) $d_1(d_1 - d_2)z_1 + d_1^2 z_2 = e^{x+y}$

$d_1 d_2 z_1 + d_1(d_1 - d_2)z_2 = 0$

$$\left\| \begin{array}{cc|c} d_1^2 - d_1 d_2 & d_1^2 & e^{x+y} \\ d_1 d_2 & d_1^2 - d_1 d_2 & 0 \end{array} \right\|$$

$$\Delta \equiv d_1^2 \begin{vmatrix} d_1 - d_2 & d_1 \\ d_2 & d_1 - d_2 \end{vmatrix} = d_1^2(d_1^2 - 3d_1 d_2 + d_2^2)$$

$$= d_1^2\,[(d_1 - \tfrac{3}{2}d_2)^2 - \tfrac{5}{4}d_2^2]$$

$$\Delta = 0, \qquad d_1 = 0, 0, \qquad \tfrac{1}{2}(3 \pm \sqrt{5})d_2$$

$$\Delta \cdot V = 0, \qquad V = \frac{1}{\Delta}0 = f_0(y) + xf_1(y)$$

$$+ \phi_1[y + \tfrac{1}{2}(3 + \sqrt{5})x] + \phi_2[y + \tfrac{1}{2}(3 - \sqrt{5})x]$$

By second row,

$$\frac{z_1}{-d_1^2} = \frac{z_2}{d_1(d_1 - d_2)} = V$$

$$\frac{w_1}{-d_1} = \frac{w_2}{d_1 - d_2} = d_1^2 V = \frac{1}{(d_1 - \tfrac{3}{2}d_2)^2 - \tfrac{5}{4}d_2^2}0 = W$$

$$W = \phi_1[y + \tfrac{1}{2}(3 + \sqrt{5})x] + \phi_2[y + \tfrac{1}{2}(3 - \sqrt{5})x]$$

$$w_1 = -d_1 W = -\tfrac{1}{2}(3 + \sqrt{5})\phi_1' - \tfrac{1}{2}(3 - \sqrt{5})\phi_2'$$

$$w_2 = (d_1 - d_2)W = [\tfrac{1}{2}(3 + \sqrt{5}) - 1]\phi_1' + [\tfrac{1}{2}(3 - \sqrt{5}) - 1]\phi_2'$$

$$= \tfrac{1}{2}(1 + \sqrt{5})\phi_1' + \tfrac{1}{2}(1 - \sqrt{5})\phi_2'$$

$$z_1 = \frac{1}{d_1}[w_1 + 0] = -\phi_1 - \phi_2 + \phi_3(y)$$

$$z_2 = \frac{1}{d_1}[w_2 + 0] = \frac{1 + \sqrt{5}}{3 + \sqrt{5}}\phi_1 + \frac{1 - \sqrt{5}}{3 - \sqrt{5}}\phi_2 + \phi_4(y)$$

$$\Delta\bar{z}_1 = \begin{vmatrix} e^{x+y} & d_1^2 \\ 0 & d_1(d_1 - d_2) \end{vmatrix} = d_1(d_1 - d_2)e^{x+y}$$

$$\bar{z}_1 = \frac{d_1(d_1 - d_2)}{d_1^2[(d_1 - \tfrac{3}{2}d_2)^2 - \tfrac{5}{4}d_2^2]}e^{x+y} = 0$$

$$\Delta\bar{z}_2 = \begin{vmatrix} d_1(d_1 - d_2) & e^{x+y} \\ d_1 d_2 & 0 \end{vmatrix} = -d_1 d_2 e^{x+y}$$

$$\bar{z}_2 = \frac{-d_1 d_2}{d_1^2[(d_1 - \tfrac{3}{2}d_2)^2 - \tfrac{5}{4}d_2^2]}e^{x+y}$$

$$= \frac{-1}{(1 - \tfrac{3}{2})^2 - \tfrac{5}{4}}e^{x+y} = e^{x+y}$$

Thus,

$$z_1 = -\phi_1[y + \tfrac{1}{2}(3 + \sqrt{5})x] - \phi_2[y + \tfrac{1}{2}(3 - \sqrt{5})x] + \phi_3(y)$$

$$z_2 = \frac{1 + \sqrt{5}}{3 + \sqrt{5}}\phi_1\left[y + \frac{1}{2}(3 + \sqrt{5})x\right] + \frac{1 - \sqrt{5}}{3 - \sqrt{5}}\phi_2\left[y + \frac{1}{2}(3 - \sqrt{5})x\right] + \phi_4(y) + e^{x+y}$$

(*d*) $X - d_1V = 0$ [Bateman]

$Y - d_2V = 0$

$d_1X + d_2Y = 0$

$$\begin{aligned} X \qquad\qquad - d_1V &= 0 \\ Y - d_2V &= 0 \\ d_1X + d_2Y \qquad &= 0, \end{aligned} \qquad \left\|\begin{matrix} 1 & 0 & -d_1 \\ 0 & 1 & -d_2 \\ d_1 & d_2 & 0 \end{matrix}\right\|$$

$$\Delta \equiv \begin{vmatrix} 1 & 0 & -d_1 \\ 0 & 1 & -d_2 \\ d_1 & d_2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -d_2 \\ d_2 & d_1^2 \end{vmatrix} = d_1^2 + d_2^2,$$

"The algebraic eliminant"

$$\frac{X}{\begin{vmatrix} 1 & -d_2 \\ d_2 & 0 \end{vmatrix}} = \frac{Y}{\begin{vmatrix} -d_2 & 0 \\ 0 & d_1 \end{vmatrix}} = \frac{V}{\begin{vmatrix} 0 & 1 \\ d_1 & d_2 \end{vmatrix}} = W = \frac{1}{\Delta} \cdot 0$$

$$\Delta \cdot W = 0, \qquad W = \frac{1}{d_1^2 + d_2^2} \cdot 0 = \phi_1(y + ix) + \phi_2(y - ix)$$

$$\begin{aligned} X &= d_2^2(\phi_1 + \phi_2) = -\phi_1'' - \phi_2'' \\ Y &= -d_1d_2(\phi_1 + \phi_2) = -i\phi_1'' + i\phi_2'' \\ V &= -d_1(\phi_1 + \phi_2) = -i\phi_1' + i\phi_2' \end{aligned}$$

For the student:

(*e*) $\dfrac{\partial \theta}{\partial t} = \dfrac{\partial u}{\partial t}$ [Ingersoll and Zobel]

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\beta}{h^2}e^{-ax}$$

(*f*) $\dfrac{\partial V}{\partial t} = \alpha\dfrac{\partial U}{\partial x}$ [Bateman]

$$\beta\frac{\partial V}{\partial x} = \frac{\partial U}{\partial t}$$

(*g*) $\dfrac{\partial V}{\partial x} = \dfrac{\partial U}{\partial t} + RU$

$$\frac{\partial U}{\partial x} = C\frac{\partial V}{\partial t} + SV$$

(h) $\theta\dfrac{\partial V}{\partial t} - \gamma\dfrac{\partial V}{\partial x} - \mu V = \alpha\dfrac{\partial U}{\partial x} + \lambda U$ [Bateman]

$\phi\dfrac{\partial U}{\partial t} - \delta\dfrac{\partial U}{\partial x} - \sigma U = \beta\dfrac{\partial V}{\partial x} + \tau V$

$\alpha, \beta, \gamma, \delta, \phi, \theta, \lambda, \mu, \sigma, \tau$ constants

(i) $\dfrac{\partial i}{\partial x} = Ge + C\dfrac{\partial e}{\partial t}$ [Berg]

$\dfrac{\partial e}{\partial x} = Ri + L\dfrac{\partial i}{\partial t}$

(12) If $H = ad_1 + bd_2 + c$, where $a$, $b$, $c$ are real integers, not all zero, all of the following are readily solvable under the foregoing theory:

(1) $Hz_2 = z_1 - z_2$
$Hz_1 = 2z_2$

(2) $-2z_1 + (H + 1)z_2 = 0$
$(H - 6)z_1 + 5z_2 = 0$

(3) $(2H + 3)z_1 + Hz_2 = 0$
$(H + 3)z_1 - 2z_2 = 0$

(4) $Hz_1 = k_1(z_1 - z_2)$
$Hz_2 = k_2z_2$

(5) $Hz_1 + az_2 = 0$
$bz_1 + Hz_2 = 0$

(6) $(H - 3)z_1 + z_2 = 0$
$z_1 - (H - 1)z_2 = 0$

(7) $Hz_1 = a_{11}z_1 + a_{12}z_2$
$Hz_2 = a_{21}z_1 + a_{22}z_2$

(8) $(2H^2 - H + 9)z_1 - (H^2 + H + 3)z_2 = 0$
$(2H^2 + H + 7)z_1 - (H^2 - H + 5)z_2 = 0$

(9) $H^2z_1 + n^2z_2 = 0$
$H^2z_2 - n^2z_1 = 0$

(10) $H^2z_2 = z_1$
$H^2z_1 = z_2$

(11) $(H^2 - 4H)z_1 - (H - 1)z_2 = 0$
$(H + 6)z_1 + (H^2 - H)z_2 = 0$

(12) $(H^2 - 3H + 2)z_1 + (H - 1)z_2 = 0$
$-(H - 1)z_1 + (H^2 - 5H + 4)z_2 = 0$

(13) $(H^2 - 2H)z_1 - z_2 = 0$
$(2H - 1)z_1 + H^2z_2 = 0$

(14) $(H^2 + 1)z_1 + (H^2 + H + 1)z_2 = x + y$
$Hz_1 + (H + 1)z_2 = e^{x+y}$

(15) $Hz_2 + z_1 = 1$
$Hz_1 + z_2 = 1$

(16) $Hz_2 + Hz_1 + 5z_1 - 3z_2 = x + y + e^{x+y}$
$Hz_1 + 2z_1 - z_2 = e^{x+y}$

(17) $Hz_1 + z_1 = e^{x+y}$
$Hz_2 = z_1$

(18) $(H + 1)z_1 = z_2 + e^{x+y}$
$(H + 1)z_2 = z_1 + e^{x+y}$

(19) $Hz_1 = z_1 + z_2 + 2 \cos (\alpha x + \beta y)$
$Hz_2 = 3z_1 - z_2$

(20) $Hz_1 = z_1 - z_2 - (\alpha x + \beta y + 2)$
$Hz_2 = z_2 + (\alpha x + \beta y)$

(21) $(H - 6)z_1 + 16z_2 = 48 \cos 2(\alpha x + \beta y)$
$(H + 2)z_2 - z_1 = 6 \cos 2(\alpha x + \beta y) - 2 \sin 2(\alpha x + \beta y)$

(22) $(H - a)z_1 - bz_2 = f_1(\alpha x + \beta y)$
$-lz_1 + (H - m)z_2 = f_2(\alpha x + \beta y)$

(23) $H^2z_1 + nHz_2 = a \cos n(x + y)$
$-nHz_1 + H^2z_2 = 0$

(24) $z_1 + Hz_2 - Hz_3 = 0$
$-Hz_1 + z_2 + Hz_3 = 0$
$Hz_1 - Hz_2 + z_3 = 0$

(25) $Hz_1 - z_2 - z_3 = 0$
$-z_1 + Hz_2 - z_3 = 0$
$-z_1 - z_2 + Hz_3 = 0$

(26) $Hz_1 - nz_2 + mz_3 = 0$
$nz_1 + Hz_2 - lz_3 = 0$
$-mz_1 + lz_2 + Hz_3 = 0$

(27) $(H - a)z_1 = 0$
$(H - a)z_2 = z_1$
$(H - a)z_3 = z_2$

(28) $(H^2 - 1)z_1 + z_2 + z_3 = 0$
$z_1 + (H^2 - 1)z_2 + z_3 = 0$
$z_1 + z_2 + (H^2 - 1)z_3 = 0$

(29) $H^2z_1 - 2a\omega Hz_2 = 0$
$2a\omega Hz_1 + H^2z_2 + 2b\omega Hz_3 = 0$
$-2b\omega Hz_2 + H^2z_3 = 0 \quad [a^2 + b^2 = 1]$

# CHAPTER IX

## THE OPERATOR $d_i \equiv \frac{\partial}{\partial x_i}$

### §38. Fundamental Theorems.

(1) We are here defining $d_i$ as $\frac{\partial}{\partial x_i}$, so that the inverses are

$$d_i^{-1} \equiv \frac{1}{d_i} \equiv \int ()\partial x_i + \phi(x_1, x_2, \cdots x_{i-1}, x_{i+1}, \cdots, x_n)$$
$$\equiv \int_a^{x_i} ()\partial x_i, \qquad i = 1, 2, \cdots, n$$

The $d_i$ are commutative relatively to each other, as the $x_i$ are all independent; functions of them are commutative; the index law applies to them and their functions; expansions can be made in ascending or descending powers of the $d_i$—all as in the case of two independent variables.

(2) The theorems of this section are but extensions of the fundamental theorems for two independent variables and are applicable to three or many independent variables. The fundamental theorems are the following:

$$\text{I.} \qquad F(d_i)e^{\phi(x_i)} \equiv e^{\phi(x_i)}F\bigg|\, d_i + \frac{\partial \phi}{\partial x_i}$$

$$\text{II.} \qquad e^{\phi(d_i)}f(x_i) \equiv f\left[x_i + \frac{\partial \phi}{\partial d_i}\right]e^{\phi(d_i)}$$

$$\text{III.} \qquad F(\pi)\Theta_m \equiv \Theta_m F(\pi + m), \qquad \pi \equiv \sum_{i=1}^{m} a_i d_i$$

$$\text{IV.} \qquad F_k(d_i)\phi\left(\sum_i a_i x_i\right) = F_k(a_i)\phi^{(k)}\left(\sum_i a_i x_i\right),$$

$k$ the degree of homogeneity of $F$ in $d_i$

V. (a) $$F(d_i) \equiv exp\left[\sum p_i \frac{\partial}{\partial q_i}\right] \cdot F(q_i)$$

$$\vdots \sum_0^{\infty} \frac{1}{n!}\left[\sum p_i \frac{\partial}{\partial q_i}\right]^n \cdot F(q_i)$$

(b) $$\equiv exp\left[\sum_{i=1}^{n}\left(\sum_{j=1, \neq s}^{n} p_{ij}\right)\frac{\partial}{\partial p_{si}}\right] \cdot F(p_{si})$$

(3) The preceding notations are:

$F(d_i) \equiv F(d_1, d_2, d_3, \cdots, d_n)$
$\phi(x_i) \equiv \phi(x_1, x_2, x_3, \cdots, x_n)$
$F(a_i) \equiv F(a_1, a_2, \cdots, a_n)$
$\phi^{(k)}$ means $k$ successive differentiations of $\phi$ with respect to the complete argument $t \equiv \sum_i a_i x_i$

$exp[] \equiv e^{[]}$
$F(q_i) \equiv F(q_1, q_2, \qquad , q_n)$
etc.

(4) *Theorem* I can be built up by induction, as in the case of Theorem I of Chap. VII §29 (2). The generalization was suggested, but never proved, by Charles Graves in 1853.* The inverse (I*a*) likewise can be proved in the same way. Two special cases are interesting and useful.

(5) I*b*. $$F(d_i)e^{\Sigma a_i x_i} \equiv e^{\Sigma a_i x_i} F(d_i + a_i)$$

This can be obtained from the general theorem by the use of

$$\phi(x_i) \equiv \sum_i a_i x_i$$

(6) I*c*. $$F(d_i) \cdot e^{\Sigma a_i x_i} 1 = e^{\Sigma a_i x_i} F(a_i)$$

This is obtained from I*b* by the operation

$$F(d_i + a_i)1 = F(a_i)$$

using Taylor's series for $F(d_i + a_i)$.

* Graves, Charles, On the Principles which Regulate the Interchange of Symbols in Certain Symbolic Equations, *Proc. Roy. Irish Acad.* (1853), 144–152.

(7) *Theorem* II is Charles Graves's correlative to Theorem I and can also be built up by induction, as in VII §29 (8). The proofs of both I and II should be set as exercises for the student. Its inverse II*a* is true, and two special cases here are interesting and useful.

(8) II*b*. $$e^{\Sigma h_i d_i} \cdot f(x_i) \equiv f(x_i + h_i) \cdot e^{\Sigma h_i d_i}$$

where $\phi(d_i) \equiv \Sigma h_i d_i$ in the general theorem.

(9) II*c*. $$e^{\Sigma h_i d_i} \cdot f(x_i) 1 = f(x_i + h_i)$$

as $e^{\Sigma h_i d_i} 1 = 1$. This is the symbolic form of Taylor's theorem for many variables.

(10) *Theorem* III. $F(\pi) \cdot \Theta_m \equiv \Theta_m \cdot F(\pi + m)$. This theorem was first stated and proved by Robert Carmichael.* It is based upon Euler's theorem for homogeneous functions.

(11) Euler's theorem on homogeneous functions states that if $U$ is a homogeneous function of the independent variables, of degree $m$,

$$\sum_i x_i d_i U_m = m U_m$$

Since

$$\pi \equiv \sum_i x_i d_i \text{ is linear, we also have}$$

$$\begin{aligned} \pi U V &= U \pi V + V \pi U \\ &= U \pi V + V m U \\ &= U(\pi + m) V \end{aligned}$$

If

$$U \equiv \Theta_m,$$
$$\pi \cdot \Theta_m \equiv \Theta_m \cdot (\pi + m)$$

Now,

$$\begin{aligned} \pi^2 \Theta_m &\equiv \pi \Theta_m \cdot (\pi + m) \\ &\equiv \Theta_m (\pi + m)(\pi + m) \equiv \Theta_m (\pi + m)^2 \end{aligned}$$

by the iterated use of Euler's theorem.

so that

$$\pi^k \cdot \Theta_m \equiv \Theta_m (\pi + m)^k$$

* CARMICHAEL, ROBERT, Homogeneous Functions and Their Index Symbol, *Cambridge and Dublin Math. Jour.* (1852), 129–140.

Then form a rational integral function with this by first multiplying by $a_k$, a constant, and then summing for $k$, obtaining

$$\sum a_k \pi^k \Theta_m \equiv \sum \Theta_m a_k (\pi + m)^k$$

or

$$F(\pi)\Theta_m \equiv \Theta_m F(\pi + m) \tag{III}$$

(12) The inverse is also true:

$$F^{-1}(\pi)\Theta_m \equiv \Theta_m F^{-1}(\pi + m) \tag{IIIa}$$

by the operational reasoning.

(13) If we use a subject $S = 1$ with this, we have

$$F(\pi) \cdot \Theta_m \cdot 1 = \Theta_m \cdot F(\pi + m) \cdot 1$$

and by the use of Taylor's theorem

$$\begin{aligned} F(\pi + m)1 &= [F(m) + \pi F'(m) + \cdots ]1 \\ &= F(m) \end{aligned}$$

so that

$$F(\pi) \cdot \Theta_m \cdot 1 = \Theta_m \cdot F(m) \tag{IIIb}$$

(14) *Theorem* IV. $F(d_i)\phi(t) = F(a_i)\phi^{(n)}(t)$, $t = \sum_i a_i x_i$. Here the $F(d_i)$ is a homogeneous function of degree $n$ in the $d_i$. When we set $t = \sum_i a_i x_i$, we shall have

$$d_i^n \phi(t) = \frac{\partial^n}{\partial t^n}\phi(t) \cdot \left(\frac{\partial t}{\partial x_i}\right)^n = a^n \phi^{(n)}(t)$$

$$(d_i^n \cdot d_j^m \cdot d_k^p \cdots)\phi(t) = (a_i^n \cdot a_j^m \cdot a_k^p \cdots)\phi^{(m+n+p+\cdots)}(t)$$

From these an integral function can be formed which will give the theorem.

(15) With the inverse $F^{-1}$ the theorem is also true; *i.e.*,

$$F^{-1}(d_i)\phi\left(\sum_i a_i x_i\right) = F^{-1}(a_i)\phi^{(-n)}\left(\sum_i a_i x_i\right) \tag{IVa}$$

where $(-n)$ means $n$ successive integrations with respect to $t$.

(16) The inverse has an exceptional (or singular) case, *viz.*, when $F(a_i) = 0$. It is always some single or multiple linear

factor which becomes zero when the substitution is made, so the problem resolves itself into isolating that factor and considering it separately. Suppose that the factor is $(\Sigma\alpha_i d_i)^p$, and suppose that its inverse operates on $\phi\left(\sum_i a_i x_i\right)$. Then

$$\frac{}{(\Sigma\alpha_i d_i)^p} \cdot \phi(\Sigma a_i x_i) = e^{-\frac{x_j}{\alpha_j}(\Sigma\alpha_i d_i)} \cdot \frac{1}{(\alpha_j d_j)^p} \cdot e^{\frac{x_j}{\alpha_j}(\Sigma\alpha_i d_i)} \cdot \phi(\Sigma a_i x_i)$$
$$= e^{-\frac{x_j}{\alpha_j}(\Sigma\alpha_i d_i)} \frac{1}{(\alpha_j d_j)^p} \phi(\Sigma a_i x_i) \qquad i \neq j$$

and since $\phi(\Sigma a_i x_i)$ is constant with respect to $x_j$, we have

$$\frac{1}{(\alpha_j d_j)^p} \cdot 1 = \frac{1}{(\alpha_j)^p} \cdot \frac{x_j{}^p}{p!}$$

and

$$\frac{1}{(\Sigma\alpha_i d_i)^p} \phi(\Sigma a_i x_i) = e^{-\frac{x_j}{\alpha_j}(\Sigma\alpha_i d_i)} \cdot \frac{1}{(\alpha_j)^p} \cdot \frac{x_j{}^p}{p!} \cdot \phi(\Sigma a_i x_i), \qquad i \neq j$$

Now, the exponential acts as in the symbolic form of Taylor's theorem and produces the original argument

$$= \frac{1}{(\alpha_j)^p} \cdot \frac{x_j{}^p}{p!} \cdot \phi\left(\sum_i a_i x_i\right)$$

(17) It is perhaps necessary to illustrate this section by an example or two.

(*a*) $\dfrac{1}{d_1 - d_2 - d_3} \cdot (2x + y + z)$

Direct substitution of 2, 1, 1 for $d_1$, $d_2$, $d_3$ gives

$$\frac{1}{2 - 1 - 1}(2x + y + z) = \infty$$

But since

$$\frac{1}{d_1 - d_2 - d_3} \equiv e^{x(d_2+d_3)} \cdot \frac{1}{d_1} \cdot e^{-x(d_2+d_3)}$$

we have

$$\frac{1}{d_1 - d_2 - d_3}(2x + y + z) = e^{x(d_2+d_3)} \cdot \frac{1}{d_1} \cdot e^{-x(d_2+d_3)} \cdot (2x + y + z)$$
$$= e^{x(d_2+d_3)} \cdot \frac{1}{d_1}[2x + (y - x) + (z - x)]$$
$$= e^{x(d_2+d_3)}\frac{1}{d_1}(y + z)$$
$$= e^{x(d_2+d_3)}x(y + z)$$
$$= x[(y + x) + (z + x)] = x(2x + y + z)$$

(b) Similarly,

$$\frac{}{(d_1 - d_2 - d_3)^3}(2x + y + z) = \frac{x^3}{3!}(2x + y + z)$$

(c) $$\frac{1}{(d_1 - d_2 - d_3)^2}\log(2x - y + 3z) = \frac{x^2}{2}\log(2x - y + 3z)$$

while

$$\frac{}{(d_1 - d_2 - d_3)^2}\log(x + y + z) = \frac{1}{d^2}\log t = \frac{t^2}{4}(2\log t - 3)$$
$$= \tfrac{1}{4}(x + y + z)^2[2\log(x + y + z) - 3]$$

**§39. Applications.**

(1) Perhaps the most interesting applications of the foregoing theory are found in partial linear differential equations with constant coefficients for three or more independent variables. While it is not advisable within the limits of this section to go into the complete theory of these, it is to be noted that this theory parallels that for two independent variables.

(2) Therefore, only examples will be given.

(a) $(d_1^2d_2 - 2d_1d_2^2 - 3d_1^2d_3 - 3d_1d_3^2 - 2d_2^2d_3 + 6d_2d_3^2 + 7d_1d_2d_3)u = 0$ [Forsythe]

*Ans.* $u = F(x - z, y) + G(2x - y, z) + H(x, 3y + z)$.

(b) $(d_1^2 + d_2^2 + d_3^2)V = -4\pi\rho$, Poisson's Equation

(c) $(d_1 + d_2 + d_3)u = xyz$ [Carr]

(d) $(ad_1 + bd_2 + cd_3)u = x^my^nz^p$ [Carr]

(e) $(d_1^3 + d_2^3 + d_3^3 - 3d_1d_2d_3)u = x^3 + y^3 + z^3 - 3xyz$ [Forsythe]

(f) $(d_1^2 + d_1d_3 - d_2^2 - d_2d_3)u = xyz$ [Forsythe]

# CHAPTER X

## THE NONCOMMUTATIVE OPERATOR $xD \equiv \vartheta$

### §40. Definitions and Fundamental Theorems.

(1) *Definitions.* We establish $\vartheta$ as a linear differential operator of the same nature as $D$ and relate the two in the following manner:

$$D \equiv \frac{d}{dx}, \qquad\qquad \frac{\quad}{dz}$$

If $x = e^z$, and $s = s(x)$, we may write

$$\frac{ds}{dz} = \frac{ds}{dx} \cdot \frac{dx}{dz} = \frac{dx}{dz} \cdot \frac{ds}{dx}$$

and then

$$\frac{ds}{dz} = e^z \frac{ds}{dx}$$

Symbolically,

$$\vartheta \cdot s = x \cdot D \cdot s$$

Abstracting the subject $s$, we have the identity of operators

$$\vartheta \equiv x \cdot D \qquad [\text{where } x = e^z] \tag{A}$$

(2) Identity $(A)$ is a special case of a general substitution theorem

$$x = f(z); \qquad \text{inverse } z = \phi(x); \qquad s = s(x)$$

Then

$$\frac{ds}{dz} = \frac{ds}{dx} \cdot \frac{dx}{dz} = f'(z) \frac{ds}{dx}$$

giving

$$\vartheta \equiv f'[\phi(x)] \cdot D$$

(3) Before proceeding with $\vartheta$, let us first establish a simple theorem in the noncommutative operators $x$ and $D$. This theorem was first stated, proved, and used by Charles Graves in 1853 [see Appendix III, §62]. It is

$$D \cdot x \equiv x \cdot D + 1 \tag{B}$$

for

$$\begin{aligned} D(x \quad s) &= xDs + s(Dx) \\ &= xDs + s \\ &= (xD + 1)s \end{aligned}$$

whence, abstracting the subject, we have the identity.

(4) *Fundamental Theorems.* Equation ($A$) is the defining equation for the relation between $x$, $D$, and $\vartheta$; Eq. ($B$) is the elementary relation between $x$ and $D$.

(5) Now, since $\vartheta$ is a linear differential operator, it is obvious that all the fundamental theorems for ($D$, $x$) apply for ($\vartheta$, $z$). All we have to do to use them here is to put $\vartheta$ for $D$ and $z$ for $x$. Chapter II in its entirety then can be inserted here. Consider it done.

(6) When, however, we use the substitution equation $x = e^z$ in the function operated upon or used in compound operators in connection with $\vartheta$, we have an interesting series of theorems in ($\vartheta$, $x$) in addition to the set in ($\vartheta$, $z$). This new set will be as follows (VII–X):

I–VI. The set in ($\vartheta$, $z$) [II §5 (1)]

VII. $$\vartheta^n \equiv \sum_{k=n}^{-} A_k x^k D^k$$

VIII. $$x^n D^n \equiv \prod_{\alpha=0}^{n-1} (\vartheta - \alpha) \equiv \sum_{k=n}^{1} B_k \vartheta^k$$

IX. $$F(\vartheta)x^m \equiv x^m F(\vartheta + m)$$

X. $$F(\vartheta)\phi(x) \equiv \sum_{k=0}^{\infty} \frac{1}{k!}[F^{(k)}(\vartheta) \cdot \phi(x)]\vartheta^k$$

(7) *Theorem* VII. Use definition ($A$) by iteration, *i.e.*

$$\vartheta \equiv xD \tag{a}$$
$$\vartheta^2 \equiv xDxD \equiv x(Dx)D \tag{b}$$

Then, using Theorem ($B$),

$$\equiv x(xD + 1)D \equiv x^2D^2 + xD \qquad (c)$$

Again,

$$\vartheta^3 \equiv xD(x^2D^2 + xD) \equiv x(Dx^2)D^2 + x(Dx)D \qquad (d)$$

Here

$$Dx^2s = x^2Ds + s(Dx^2) = x^2Ds + 2xs = (x^2D + 2x)s$$

from which, by abstraction,

$$Dx^2 \equiv x^2D + 2x$$

which we shall substitute in ($d$), obtaining

$$\begin{aligned} \vartheta^3 &\equiv x(x^2D + 2x)D^2 + x^2D^2 + xD \\ &\equiv x^3D^3 + 3x^2D^2 + xD \end{aligned}$$

By an iterated use of this method we can obtain the theorem in the form stated, where the $A_k$ are Stirling numbers obtained successively according to the table

| $m$ | 1 | 2 | 3 | 4 | 5 | | $k$ | $k+1$ |
|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 1 | | | | | | |
| 3 | 1 | 3 | 1 | | | | | |
| 4 | 1 | 6 | 7 | 1 | | | | |
| 5 | 1 | 10 | 25 | 15 | 1 | | | |
| $m$ | .. | .. | .. | .. | .. | .. | $a_{mk}$ | $a_{m(k+1)}$ |
| $m+1$ | .. | .. | .. | .. | .. | .. | .... | $(m-k+1)a_{mk} + a_{m(k+1)}$ |

(8) *Theorem* VIII. By the use of the equations obtained for Theorem VII, we can build up as follows:

$$\begin{aligned} \vartheta &\equiv xD \\ \vartheta^2 &\equiv x^2D^2 + xD \equiv x^2D^2 + \vartheta \end{aligned}$$

or

$$\begin{aligned} x^2D^2 &\equiv \vartheta^2 - \vartheta \equiv \vartheta(\vartheta - 1) \\ \vartheta^3 &\equiv x^3D^3 + 3x^2D^2 + xD \\ &\equiv x^3D^3 + 3\vartheta(\vartheta - 1) + \vartheta \end{aligned}$$

or

$$\begin{aligned} x^3D^3 &\equiv \vartheta^3 - 3\vartheta(\vartheta - 1) + \vartheta \\ &\equiv \vartheta(\vartheta - 1)(\vartheta - 2) \end{aligned}$$

Continuing, we obtain the theorem

$$x^nD^n \equiv \prod_{\alpha=0}^{n-1} (\vartheta - \alpha) \equiv \sum_{k=n}^{1} B_k\vartheta^k$$

The $B_k$ here are similar to the $A_k$ of Theorem VII, except that the general term is $-ma_{mk} + a_{m(k+1)}$. The table of their values is

| $m$ | 1 | 2 | 3 | 4 | 5 | $k$ | $k+1$ |
|---|---|---|---|---|---|---|---|
| 2 | 1 | $-1$ | | | | | |
| 3 | 1 | $-3$ | 2 | | | | |
| 4 | 1 | $-6$ | 11 | $-6$ | | | |
| 5 | 1 | $-10$ | 35 | $-50$ | 24 | | |
| $m$ | 1 | .... | .. | .... | .. | $a_{mk}$ | $a_{m(k+1)}$ |
| $m+1$ | 1 | .... | .. | .... | .. | .... | $-ma_{mk} + a_{m(k+1)}$ |

(9) *Theorem* IX. Use Theorem I of II §5 (1) with $(\vartheta, z)$:

$$F(\vartheta)e^{\phi(z)} \equiv e^{\phi(z)}F[\vartheta + \phi'(z)]$$

With $\phi(z) = mz$:

$$F(\vartheta)e^{mz} \equiv e^{mz}F(\vartheta + m)$$

Then $e^z = x$ gives

$$F(\vartheta)x^m \equiv x^mF(\vartheta + m) \qquad \text{(IX)}$$

(10) The theorem is true for the inverse:

$$F^{-1}(\vartheta)x^m \equiv x^mF^{-1}(\vartheta + m) \qquad \text{(IX}a\text{)}$$

(11) A special case, when the subject $S = 1$, is

$$F(\vartheta)x^m \cdot 1 = x^mF(m) \qquad \text{(IX}b\text{)}$$

as can be proved by noting that $F(\vartheta + m)1 = F(m)$.

(12) *Theorem* X. Using Theorem VI of II §5 (1) with $D \approx \vartheta \equiv \vartheta_1 + \vartheta_2$, where $\vartheta_1 \sim s$ and $\vartheta_2 \sim \phi(x)$, we have

$$F(\vartheta) \equiv \sum_{k=0}^{\infty} \frac{1}{k!}\vartheta_1{}^k \cdot F^{(k)}(\vartheta_2)$$

and

$$F(\vartheta)\phi(x) \equiv \sum_{k=0}^{\infty} \frac{1}{k!}[F^{(k)}(\vartheta) \cdot \phi(x)]\vartheta^k \qquad \text{(X)}$$

(13) *Examples:*
Form products in $\vartheta$ for the following:

(a) $x\vartheta$
(b) $(x\vartheta)^2$
(c) $(x\vartheta)^m$
(d) $x^2\vartheta$
(e) $(x^2\vartheta)^2$
(f) $(x^2\vartheta)^m$
(g) $x\vartheta^2$
(h) $(x\vartheta^2)^2$
(i) $x^2\vartheta^2$
(j) $(x^2\vartheta^2)^2$
(k) $(x^2\vartheta^2)^m$

Form a series identity for each of the following:

(l) $e^{k\vartheta}$
(m) $e^{x\vartheta}$
(n) $e^{x^2\vartheta}$
(o) $\dfrac{1}{1 - x\vartheta}$
(p) $\dfrac{1}{1 - x^2\vartheta}$
(q) $\dfrac{1}{1 + x\vartheta}$
(r) $\sin \vartheta$
(s) $\cos x\vartheta$

By Leibnitz's extension expand the operators

(t) $\vartheta^2x^2$ (u) $\dfrac{1}{\vartheta^2}x^2$ (v) $\dfrac{1}{\vartheta - 1}$

Show that

(w) $e^{k\vartheta}f(x) = f(xe^k)$

(x) $e^{kx\vartheta}f(x) = f\left(\dfrac{x}{1 - nx}\right)$ [Crofton]

**§41. The Euler Equation and Its Extension.**

(1) "Euler's equation" is the name given to one of the type

$$\sum_{k=n}^{v} A_k x^k D^k \cdot y = f(x)$$

By the substitution $x = e^z$ and Theorem VIII of §40 this equation becomes

$$\sum_{k=n}^{0} A_k \left[ \prod_{\alpha=0}^{k-1} (\vartheta - \alpha) \right] \cdot y = f(e^z)$$

which is a linear differential equation with constant coefficients with $z$ as the independent variable.

(2) *Examples:*

(a) $$v\frac{dp}{dv} = -\gamma\rho$$ [Lamb]

(b) $$\frac{d^2u}{dr^2} + \frac{du}{dr} - \frac{u}{r} + ar^2 = 0$$ [Mellor]

(c) $$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = KR$$

(d) $$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} = 0$$ [Forsythe]

(e)
$$\begin{aligned}
(xD^2 + D)y &= 0 \\
(x^2D^2 + xD)y &= 0 \\
[\vartheta(\vartheta - 1) + \vartheta]y &= 0 \\
\vartheta^2 y &= 0 \\
y &= \frac{1}{\vartheta^2}0 = C_1 + C_2 z \\
&= C_1 + C_2 \log x
\end{aligned}$$

(f)
$$\begin{aligned}
(x^2D^2 + 3xD + 1)y &= 0 \\
[\vartheta(\vartheta - 1) + 3\vartheta + 1]y &= 0 \\
(\vartheta^2 + 2\vartheta + 1)y &= 0 \\
(\vartheta + 1)^2 y &= 0 \\
y &= \frac{1}{(\vartheta + 1)^2}0 = e^{-z}(C_1 + C_2 z) \\
&= x^{-1}(C_1 + C_2 \log x) \\
xy &= C_1 + C_2 \log x
\end{aligned}$$

(g)
$$\begin{aligned}
\left(D^3 - \frac{6}{x^3}\right)y &= 0 \\
(x^3D^3 - 6)y &= 0 \\
[\vartheta(\vartheta - 1)(\vartheta - 2) - 6]y &= 0 \\
(\vartheta^3 - 3\vartheta^2 + 2\vartheta - 6)y &= 0 \\
(\vartheta^2 + 2)(\vartheta - 3)y &= 0
\end{aligned}$$

$$\begin{aligned}
y &= \frac{1}{(\vartheta^2 + 2)(\vartheta - 3)}0 = C_1 e^{3z} + C_2 \cos\sqrt{2}z + C_3 \sin\sqrt{2}z \\
&= C_1 x^3 + C_2 \cos\sqrt{2}\log x + C_3 \sin\sqrt{2}\log x.
\end{aligned}$$

(h) $(x^3D^3 + 4x^2D^2 + xD - 1)y = 0$

(i) $(x^3D^3 - 3x^2D^2 + 7xD - 8)y = 0$

(j) $(x^4D^4 - \lambda D^2)y = 0$

(k) $(x^3D^3 + 6x^2D^2 + 4xD - 4)y = 0$

(l) $(xD + 1)y = x^2$

(m) $x^2D^2y = \log x$
(n) $(x^2D^2 + xD)y = 12x^{-1} \log x$
(o) $(x^2D^2 + xD + n^2)u = x^m$
(p) $(x^2D^2 + 2xD - 2)y = x \cos x - \sin x$
(q) $(x^2D^2 - 3xD + 4)y = x^m$
(r) $(x^2D^2 + 4xD + 2)y = e^x$
(s) $[x^2D^2 - (2m - 1)xD + (m^2 + n^2)]y = 0$
(t) $(x^3D^3 + xD - 1)y = x \log x$

(3) An **extension** of Euler's equation would be

$$\sum_{k=n}^{0} A_k(\alpha + \beta x)^k D^k \cdot y = f(x)$$

which by the substitution $(\alpha + \beta x) = e^z$ reduces to a linear equation with constant coefficients:

$$\sum_{k=n}^{0} A_k \prod_{\alpha=0}^{n-1} (\vartheta - \alpha) \cdot y = f\left(\frac{e^z - \alpha}{\beta}\right) = F(e^z)$$

(4) *Examples:*

(a)
$$[(a + x)^2D^2 - 5(a + x)D + 6]y = 0$$
$$[\vartheta(\vartheta - 1) - 6\vartheta + 6]y = 0$$
$$(\vartheta^2 - 7\vartheta + 6)y = 0$$
$$(\vartheta - 6)(\vartheta + 1)y = 0$$
$$y = \frac{1}{(\vartheta - 6)(\vartheta + 1)}0 = C_1e^{6z} + C_2e^{-z}$$
$$= C_1(a + x)^6 + C_2(a + x)^{-1}$$

(b) $[(x + 1)^3D^3 + (x + 1)^2D^2 + 3(x + 1)D - 8]y = 0$
(c) $[(x + 1)^2D^2 + (x + 1)D + 1]y = 4 \cos \log (x + 1)$

(d)
$$t\frac{dx}{dt} + y = 0$$ [Piaggio]
$$t\frac{dy}{dt} + x = 0$$
$$t = e^z, \qquad tD \equiv \vartheta$$
$$\vartheta x + y = 0$$
$$x + \vartheta y = 0$$
$$\Delta = \qquad = \vartheta^2 - 1 = (\vartheta + 1)(\vartheta - 1)$$

$$\Delta \cdot V = 0, \qquad V = \frac{1}{\Delta}0 = \frac{1}{(\vartheta+1)(\vartheta-1)}0 = C_1e^{-z} + C_2e^{z}$$

$$\frac{x}{\vartheta} = \frac{y}{-1} = V$$

$$x = \vartheta(C_1e^{-z} + C_2e^{z}) = -C_1e^{-z} + C_2e^{z}$$

$$y = -C_1e^{-z} - C_2e^{-z}$$

$$\begin{cases} x = At + Bt^{-1} \\ y = Bt^{-1} - At \end{cases}$$

(e) $$(t^2D^2 + tD)x + 2y = 0$$ [Piaggio]

$$(t^2D^2 + tD)y - 2x = 0$$

# CHAPTER XI

## SOLUTIONS IN SERIES

### §42. Linear Differential Equations.

(1) There is a large class of differential equations which offer considerable difficulty to the ordinary worker in science, because their solutions are complicated series which require considerable advanced mathematical analysis for their study and proper use. Even to obtain the solutions is usually a complicated process. We are going to show that the operational method, with the $\vartheta$ operator treated in the last chapter, offers an easily acquired method of obtaining with surety any polynomial or infinite series solution of any of the class of *ordinary linear differential equations with polynomial coefficients.* We shall first exhibit a familiar equation and apply the $\vartheta$ operator to it, *viz.*, the Bessel equation of order 2. Bessel equations have many applications in modern engineering work.*

(2) *The Bessel Equation of Order* 2.

The differential equation:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 2^2)y = 0$$

The operational form:

$$(x^2D^2 + xD + x^2 - 2^2)y = 0 \qquad (x,\ D) \text{ form}$$

with the substitution $x = e^z$, we have $xD \equiv \vartheta$, $x^2D^2 \equiv \vartheta^2 - \vartheta$, and the equation becomes

$$[(\vartheta^2 - 2^2) + e^{2z}]y = 0 \qquad (z,\ \vartheta) \text{ form}$$

Operate on both sides by $(\vartheta^2 - 2^2)^{-1}$, obtaining

$$[1 + (\vartheta^2 - 2^2)^{-1} \cdot e^{2z}]y = (\vartheta^2 - 2^2)^{-1} \cdot 0$$

Set $(\vartheta^2 - 2^2)^{-1} \cdot e^{2z} \equiv S$ on the left, and perform the operation on the right:

See McLachlan, N. W., "Bessel Functions for Engineers," Oxford, 1934.

$$(1 + S)y = C_1e^{2z} + C_2e^{-2z}$$
$$y = \frac{1}{1 + S}(C_1e^{2z} + C_2e^{-2z})$$

Now, algebraically, by actual division,

$$\frac{1}{1 + S} \equiv \sum_{k=0}^{\infty} (-1)^k S^k$$

Thus,

$$= \sum_{k=0}^{\infty} (-1)^k S^k \cdot (C_1e^{2z} + C_2e^{-2z}) \tag{a}$$

(3) Each $S^k$ operating on the binomial $C_1e^{2z} + C_2e^{-2z}$ gives us a term of two series which are the solution of the $\vartheta$ equation and thus of the original equation. It is sufficient to use only the general $k$ for this purpose, which gives the general term of each series; but, that the process may be seen in its details, we shall use each successive $k$ starting with $k = 0$, until the form of the successive results is seen. We shall also take each term of the subject separately.

(4) $$S^0 \equiv 1, \qquad S^0 \cdot C_1e^{2z} = C_1e^{2z}$$

$$S^1 \equiv \frac{1}{\vartheta^2 - 2^2}e^{2z} \equiv e^{2z}\frac{1}{(\vartheta + 2)^2 - 2^2} \quad \text{by shifting theorem}$$

$$S^1 \cdot C_1e^{2z} = e^{2z}\frac{1}{(\vartheta + 2)^2 - 2^2}C_1e^{2z}$$
$$= C_1e^{4z}\frac{1}{(2 + 2)^2 - 2^2} \qquad \text{by substitution}$$
$$= C_1e^{(2+2)z}\frac{1}{4 \cdot 3 \cdot 1}$$

$$S^2 \equiv \frac{1}{\vartheta^2 - 2^2}e^{2z} \cdot \frac{1}{\vartheta^2 - 2^2}$$
$$e^{2(2z)}\frac{1}{(\vartheta + 2 \cdot 2)^2 - 2^2} \cdot \frac{}{(\vartheta + 2)^2 - 2^2}$$

$$S^2 \cdot C_1e^{2z} = e^{2(2z)}\frac{1}{(\vartheta + 2 \cdot 2)^2 - 2^2} \cdot \frac{1}{(\vartheta + 2)^2 - 2^2}C_1e^{2z}$$
$$= C_1e^{(2 \cdot 2+2)z}\frac{1}{(2 + 2 \cdot 2)^2 - 2^2} \cdot \frac{1}{(2 + 2)^2 - 2^2}$$
$$C_1e^{(2 \cdot 2+2)z}\frac{1}{4 \cdot 4 \cdot 2} \cdot \frac{1}{4 \cdot 3 \cdot 1}$$

$$S^3 \equiv \frac{1}{\vartheta^2 - 2^2} e^{2z} \cdot \frac{1}{\vartheta^2 - 2^2} e^{2z} \cdot \frac{1}{\vartheta^2 - 2^2} e^{2z}$$

$$= e^{\cdots} \frac{1}{(\vartheta + 3 \cdot 2)^2 - 2^2} \cdot \frac{1}{(\vartheta + 2 \cdot 2)^2 - 2^2} \cdot \frac{1}{(\vartheta + 2)^2 - 2^2}$$

$$S^3 \cdot C_1 e^{2z} = C_1 e^{(3 \cdot 2 + 2)z} \frac{1}{(2 + 3 \cdot 2)^2 - 2^2} \cdot \frac{1}{(2 + 2 \cdot 2)^2 - 2^2} \cdot \frac{1}{(2 + 2)^2 - 2^2}$$

$$= C_1 e^{(3 \cdot 2 + 2)z} \frac{1}{4 \cdot 5 \cdot 3} \cdot \frac{1}{4 \cdot 4 \cdot 2} \cdot \frac{1}{4 \cdot 3 \cdot 1}$$

Similarly,

$$S^4 \cdot C_1 e^{2z} = C_1 e^{(4 \cdot 2 + 2)z} \frac{1}{4 \cdot 6 \cdot 4} \cdot \frac{1}{4 \quad 5 \cdot 3} \cdot \frac{1}{4 \cdot 4 \cdot 2} \cdot \frac{1}{4 \cdot 3 \cdot 1}$$

or

$$S^k \cdot C_1 e^{2z} = C_1 e^{(k \cdot 2 + 2)z} \prod_{h=1}^{k} \frac{1}{(2 + h \cdot 2)^2 - 2^2}$$

$$= C_1 e^{(k \cdot 2 + 2)z} \prod_{h=1}^{k} \frac{1}{4 \cdot (h + 2)h}$$

Summing for $k$, we have for the first series

$$C_1 e^{2z} \left[ 1 + \sum_{k=1}^{\infty} (-1)^k e^{2kz} \prod_{h=1}^{k} \frac{1}{4(h + 2)h} \right.$$

or

$$C_1 x^2 \left[ 1 + \sum_{k=1}^{\infty} (-1)^k x^{2k} \prod_{h=1}^{k} \frac{1}{4(h + 2)h} \right]$$

(5) The operation on the second term of the binomial brings in quite another type of series; *e.g.*,

$$S^0 C_2 e^{-2z} = C_2 e^{-2z}$$

$$S^1 C_2 e^{-2z} = C_2 e^{(2-2)z} \frac{1}{(-2 + 2)^2 - 2^2} = C_2 e^{(2-2)z} \frac{1}{-2^2}$$

$$S^2 C_2 e^{-2} \qquad e^{2(2z)} \frac{1}{(\vartheta + 2 \cdot 2)^2 - 2^2} \cdot \frac{1}{(\vartheta + 2)^2 - 2^2} \cdot C_2 e^{-2z}$$

Here the operational procedure changes, for if we merely substitute $-2$ for $\vartheta$ in the second factor $\frac{1}{(\vartheta + 2 \cdot 2)^2 - 2^2}$, we shall obtain an infinity. We must again use the shifting theorem for it and the substitution theorem for the first, obtaining

$$C_2 e^{(2\cdot 2-2)z}\frac{1}{(\vartheta - 2 + 2\cdot 2)^2 - 2^2}\cdot\frac{1}{-2^2}$$

The denominator of the second factor is

$$(\vartheta - 2 + 2\cdot 2)^2 - 2^2 = \vartheta^2 + 4\vartheta + 4 - 4 = \vartheta^2 + 4\vartheta$$
$$= \vartheta(\vartheta + 2\cdot 2)$$

Now, since $\frac{1}{-2^2}$ is constant,

$$\frac{1}{\vartheta + 2\cdot 2}\cdot\left(\frac{1}{-2^2}\right) = \frac{1}{2\cdot 2}\left(\frac{1}{-2^2}\right)$$

then

$$\frac{1}{\vartheta}\frac{1}{2\cdot 2}\left(\frac{1}{-2^2}\right) = \frac{z}{2\cdot 2}\left(\frac{1}{-2^2}\right)$$

we thus have

$$S^2 C_2 e^{-2z} = C_2 e^{(2\cdot 2-2)z}\frac{z}{2\cdot 2}\cdot\frac{1}{-2^2}$$

This $\frac{z}{2\cdot 2}$ will now appear in the result of every operation after this. Witness

$$S^3 C_2 e^{-2z} =$$
$$e^{3(2z)}\frac{1}{(\vartheta + 3\cdot 2)^2 - 2^2}\cdot\frac{1}{(\vartheta + 2\cdot 2)^2 - 2^2}\cdot\frac{}{(\vartheta + 2)^2}\quad\frac{}{2^2}C_2 e^{-2z}$$

For the first and third operations we can substitute $-2$ for $\vartheta$. For the second, we only shift across and integrate as above, obtaining

$$S^3 C_2 e^{-2z} = C_2 e^{(3\cdot 2-2)z}\frac{z}{2\cdot 2}\cdot\frac{1}{(-2 + 3\cdot 2)^2 - 2^2}\cdot\frac{1}{(-2 + 2)^2 - 2^2}$$
$$= C_2 e^{(3\cdot 2-2)z}\frac{z}{2\cdot 2}\cdot\frac{1}{2^2(2^2 - 1)}\cdot\frac{1}{2^2(0^2 - 1)}$$

Then for the $k$th term it is easily seen that

$$S^k C_2 e^{-2z} = C_2 e^{(2k-2)z}\frac{z}{2\cdot 2}\prod_{\substack{h=1\\ \neq 2}}^{k}\frac{1}{(2h - 2)^2 - 2^2}$$

Now, for the summation we have

$$C_2 e^{-2z}\left[1 + \frac{1}{2^2}e^{2z} + \frac{z}{2\cdot 2}\sum_{k=2}^{\infty}(-1)^k e^{k(2z)}\prod_{\substack{h=1\\ \neq 2}}^{k}\frac{1}{2^2 h(h - 2)}\right.$$

or

$$C_2 x^{-2} \; 1 + \frac{1}{2^2}x^2 + \frac{1}{2 \cdot 2} \log x \sum_{k=2}^{\infty} (-1)^k x^{2k} \prod_{\substack{h=1 \\ \neq 2}}^{k} \frac{1}{2^2 h(h-2)}$$

These two series will be recognized as the solutions of the original equation. Their sum is the complete solution.

(6) Looking closely at $(a)$,

$$y = \sum_{k=0}^{\infty} (-1)^k S^k (C_1 e^{2z} + C_2 e^{-2z})$$

we see that it really has the form of an integral equation but in symbolic form. One could surmise that integral equation theory could therefore be applied to it, or, what is better, an integral equation could be turned over into this symbolic type, and operational theory applied for its solution. This latter we are going to do. The theory in the following sections handles all the Bessel equations as special cases. The general theory takes in all possible types of ordinary linear differential equations with polynomial coefficients.

**§43. $\vartheta$-Transformations.**

(1) *Definition.* The $\vartheta$-transformation is that which transforms a differential form in $(x, D)$ into one in $(z, \vartheta)$. For this purpose, we shall use the following:

$(a)$ $x = e^z$

$(b)$ $xD \equiv \vartheta$ [see X §40 (1) $(A)$]

$(c)$ $x^n D^n \equiv \prod_{\alpha=0}^{n-1} (\vartheta - \alpha)$ [X §40 (6) (VIII)]

(2) Consider a form $x^m D^n$, where $m \gtreqless n$.

$(a)$ $m > n$; say, $m - h = n$. We shall have

$$x^m D^n \equiv x^h x^n D^n$$

from which by $(a)$ and $(c)$ above we obtain

$$x^h (x^n D^n) \quad e^h \quad \overset{n-1}{\underset{0}{\phantom{x}}} \; (\vartheta - \underline{\alpha})$$

we are using $\left| \, n - \underset{0}{1} \, \right|$ for $\prod_{\alpha=0}^{n-1}$ as a simpler form, putting a bar under the ranging letter].

(*b*) $m = n$. We have, by (*c*),

$$x^nD^n = \Big|_{\,0}^{\,n-1} (\vartheta - \underline{\alpha})$$

(*c*) $m < n$; say, $m + h = n$. If we multiply the entire differential form through by $x^h$, then all terms will come under either (*a*) or (*b*).

(3) All forms $\Big|_{\,0}^{\,n-1} (\vartheta - \underline{\alpha})$ can be expanded into powers of $\vartheta$ as shown in Chap. X §40 (8), *i.e.*, with $B_k$ coefficients. We repeat the table but include the coefficients up to $m$, $k = 9$.

| $m = 1$ | $k = 1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | $-1$ | | | | | | | |
| 3 | 1 | $-3$ | 2 | | | | | | |
| 4 | 1 | $-6$ | 11 | $-6$ | | | | | |
| 5 | 1 | $-10$ | 35 | $-50$ | 24 | | | | |
| 6 | 1 | $-15$ | 85 | $-325$ | 274 | $-120$ | | | |
| 7 | 1 | $-21$ | 175 | $-835$ | 2224 | $-1764$ | 720 | | |
| 8 | 1 | $-28$ | 322 | $-2060$ | 8069 | $-17332$ | 13068 | $-5040$ | |
| 9 | 1 | $-36$ | 546 | $-4636$ | 24549 | $-81884$ | 151724 | $-109584$ | 40320 |

| | $k$ | $k + 1$ |
|---|---|---|
| $m$ | $a_{mk}$ | $a_{m,k+1}$ |
| $m + 1$ | ........ | $-ma_{m,k} + a_{m,k+1}$ |

Then we may collect terms with like powers of the exponential $e^z$. We shall then have the proper form in $(z, \vartheta)$.

(4) The $(z, \vartheta)$ form is found to be of the general type

$$[f_0(\vartheta)e^{k_0 z} + f_1(\vartheta)e^{k_1 z} + \qquad + f_n(\vartheta)e^{k_n z}]y = 0$$

or

$$\sum_{j=0} f_j(\vartheta)e^{k_j z} \cdot y = 0$$

The $k_j$ will be positive or negative integers. We may divide through on the right by $e^{k_0 z}$ and obtain the form where $k_0 = 0$. The $k_j$ may then run consecutively from 0 to $n$ or be a set of arbitrary integers.

**§44. Classification.**

(1) The $(z, \vartheta)$ form offers a convenient method of classifying our linear equations which, so far as the author can find, has never been attempted in complete form before. Classification on the basis of the poles of polynomial coefficients in the $(x, D)$ form has been suggested by Whittaker and Watson, Ince, Piaggio, Bateman, etc.* The classification here given is based on the poles of the $(z, \vartheta)$ form and in terms of the roots of the *$\vartheta$-polynomial z-exponential* coefficients of the independent variable. These roots and coefficients of $z$ enter into the solutions in vital ways in determining the types, poles, zeros, etc., of the solutions. In fact, they determine the complete pattern of the solutions.

(2) We may show this method of classification most quickly by an example, leading up to the general classification symbol. Take the Legendre equation, for example. Its $(z, \vartheta)$ form is

$$[\vartheta(\vartheta - 1 + \gamma) - (\vartheta + \alpha - 1)(\vartheta + \beta - 1)e^z]y = 0$$

Here

$$f_0(\vartheta) \equiv \vartheta(\vartheta - 1 + \gamma), \qquad \text{with roots } 0,\ 1 - \gamma$$
$$f_1(\vartheta) \equiv (\vartheta + \alpha - 1)(\vartheta + \beta - 1) \qquad \text{with roots } 1 - \alpha,\ 1 - \beta$$

and

$$k_0 = 0, \qquad k_1 = 1$$

These six numbers may be written down in the bracket

$$[0, 1 - \gamma \mid 0 \parallel 1 - \alpha, 1 - \beta \mid 1]$$

Note that $f_0(\vartheta)$ has two factors and also that $f_1(\vartheta)$ has 2. In terms of the number of factors of each, we could call all equations having similar numbers of factors so situated as of the class (2, 2), leaving the $k_j$ out of consideration.

(3) The Jacobi equation has the $(z, \vartheta)$ form

$$[\vartheta(\vartheta - 1) + (\beta - \alpha)(\vartheta - 1)e^z - (\vartheta + a + b)(\vartheta + a - b)e^{2z}]y = 0$$

where

$$f_0(\vartheta) \equiv \vartheta(\vartheta - 1),$$
$$f_1(\vartheta) \equiv \vartheta - 1, \qquad f_2(\vartheta) \equiv (\vartheta + a + b)(\vartheta + a - b)$$

* WHITTAKER and WATSON, "Modern Analysis," Chap. X, p. 204. INCE, "Ordinary Differential Equations," XV, p. 393, XVI.

We may therefore put the equation in class 5 with partition (distribution) (2, 1, 2).

(4) It would be an easy matter to write down all the classes and partitions of equations by merely writing down all partitions of all integers from 1 up. Thus the two equations above:

| | Class $j$ | Partition $p$ |
|---|---|---|
| Legendre. | 4 | (2, 2) |
| Jacobi.... | 5 | (2, 1, 2) |

(5) By this means we can, if we desire, write down the type equations in general terms; thus, the Jacobi equation belongs to the type form

$$[a_0(\vartheta - \alpha_{01})(\vartheta - \alpha_{02}) + a_1(\vartheta - \alpha_{11})e^{k_1 z} + a_2(\vartheta - \alpha_{21})(\vartheta - \alpha_{22})e^{k_2 z}]y = 0$$

where the constants are

$$a_0,\ a_1,\ a_2; \qquad \alpha_{01},\ \alpha_{02}; \qquad \alpha_{11}; \qquad \alpha_{21},\ \alpha_{22}; \qquad k_1,\ k_2$$

By the inverse transformation from the $(z, \vartheta)$ to the $(x, D)$ form we could obtain the ordinary type form. This is not necessary, because the polynomial coefficients do not directly point to the pattern of the solutions, as do the constants $a$, $\alpha$, $k$.

(6) To see that the $\alpha^s$ and $k^s$ do point the pattern, let us look at the Bessel equation solved in §42.

$$(x^2D^2 + xD + x^2 - 2^2)y = 0$$

$$[(\vartheta^2 - 2^2) + e^{2z}]y = 0$$

$$f_0(\vartheta) \equiv (\vartheta + 2)(\vartheta - 2), \qquad f_1(\vartheta) \equiv 1$$

$$\alpha_{01} = -2, \qquad \alpha_{02} = 2; \qquad k_0 = 0, \qquad k_1 = 2$$

$$y = C_1x^2\ 1 + \sum_{k=1} (-1)^k x^{2k} \quad k \quad (2h + 2)^2 - 2^2$$

$$+ C_2x^{-2}\ 1 + \frac{1}{2^2}x^2 + \frac{\log x}{2^2}\sum_{k=2} (-1)^k \quad \overset{k}{\underset{1}{\neq 2}} \frac{1}{(2h - 2)^2 - 2^2}$$

Note the pattern of the solutions. The $\alpha_{01}$ and $\alpha_{02}$, respectively $-2$, 2, determine the powers of the $x$ in first terms and also are found in the denominators of the coefficients $(2h + 2)^2 - 2^2$ and $(2h - 2)^2 - 2^2$. The $\alpha_{01} = -2$ determines the pole for the $\log x$, and order 2 determines the $-2^2$ in the denominators.

(7) The class symbol thus gives the number of the critical characteristics of the solution, and the complete symbol gives

the salient features of the pattern of the solutions. Hereafter when a particular equation is listed, its classification will be referred to by partition numbers.

(8) Even a cursory examination of the actual equations treated in the literature side by side with this classification of the possibilities will show that the subject of types of functions which may be solutions of linear differential equations has barely been touched. When someone exhaustively produces all types of solutions and compares them with the series known in analysis, many new, and perhaps valuable, functions will be brought to light.

(9) The table of the classification will look like the following:

$j \equiv$ class number (sum of partitions)

$p \equiv$ the partition symbol

$i \equiv$ the number of $f_i(\vartheta)$:

| $j$ | $p$ | | | |
|---|---|---|---|---|
| | $i = 1$ | 2 | 3 | 4 |
| 1 | (1) | (0 1)<br>(1 0) | (0 0 1)<br>(0 1 0)<br>(1 0 0) | (1 0 0 0)<br>(0 1 0 0)<br>(0 0 1 0)<br>(0 0 0 1) |
| 2 | (2) | (2 0)<br>(1 1)<br>(0 2) | (2 0 0)<br>(0 2 0)<br>(0 0 2)<br>(1 1 0)<br>(1 0 1)<br>(0 1 1) | (2 0 0 0)<br>(0 2 0 0)<br>(0 0 2 0)<br>(0 0 0 2)<br>(1 1 0 0)<br>(1 0 1 0)<br>(1 0 0 1)<br>(0 1 1 0)<br>(0 1 0 1)<br>(0 0 1 1) |
| 3 | (3) | (3 0)<br>(2 1)<br>(1 2)<br>(0 3) | (3 0 0)<br>(0 3 0)<br>(0 0 3)<br>(2 1 0)<br>(2 0 1)<br>(1 2 0)<br>(1 0 2)<br>(0 1 2)<br>(0 2 1) | Etc. |
| Etc. | | | | |

**§45. Fundamental Theorems.**

In this section, we shall give in detail four theorems upon which we shall base the solution of the general linear differential equation, *i.e.*,

I. The reciprocal theorem.
II. The operators $F$.
III. Max Mason's theorem.
IV. Boole's theorem.

Let us proceed at once to these.

I. *The Reciprocal Transformation.* (1) We need first a theorem for the production of the terms of a quotient. In general, $\frac{1}{1+S}$ will not be a form containing only one operator $S$, the series in this case being a simple one. In general, we shall have the form

$$\Phi \quad 1 + \phi_1 + \phi_2 + \qquad \equiv 1 + F_1 + F_2 + \qquad \equiv F$$

Let us determine the $F_k$ in terms of the $\phi_k$.

(2) Assume $\frac{1}{\Phi} \equiv F$, or $1 \equiv \Phi \cdot F$ with $\Phi \equiv \Sigma\phi_k x^k$ and $F \quad \Sigma F_k x^k$.

Multiply and equate coefficients. Set the first terms as $\phi_0$ and $F_0$ instead of 1. Then

$$\begin{aligned}
1 &= F_0\phi_0 \\
0 &= F_0\phi_1 + F_1\phi_0 \\
0 &= F_0\phi_2 + F_1\phi_1 + F_2\phi_0 \\
0 &= F_0\phi_3 + F_1\phi_2 + F_2\phi_1 + F_3\phi_0 \\
0 &= F_0\phi_4 + F_1\phi_3 + F_2\phi_2 + F_3\phi_1 + F_4\phi_0
\end{aligned}$$

$$0 = \sum_{i=0} F_i\phi_k.$$

(3) Of course, $F_0 = \frac{1}{\phi_0}$. For $F_1$ we shall use the first two equations and solve by Cramer's rule:

$$
\begin{aligned}
F_0\phi_0 \qquad\qquad &= 1 \\
F_0\phi_1 + F_1\phi_0 &= 0
\end{aligned}
$$

$$\Delta = \begin{vmatrix} \phi_0 & \cdot \\ \phi_1 & \phi_0 \end{vmatrix} = \phi_0^2$$

$$\Delta F_1 = \begin{vmatrix} \phi_0 & 1 \\ \phi_1 & \cdot \end{vmatrix} = -\phi_1$$

$$F_1 = -\frac{1}{\phi_0^2}$$

For $F_2$, using three equations,

$$
\begin{aligned}
F_0\phi_0 \qquad\qquad\qquad\qquad &= 1 \\
F_0\phi_1 + F_1\phi_0 \qquad\qquad &= 0 \\
F_0\phi_2 + F_1\phi_1 + F_2\phi_0 &= 0
\end{aligned}
$$

$$\Delta = \begin{vmatrix} \phi_0 & \cdot & \cdot \\ \phi_1 & \phi_0 & \cdot \\ \phi_2 & \phi_1 & \phi_0 \end{vmatrix} = \phi_0^3$$

$$\Delta F_2 = \begin{vmatrix} \phi_0 & \cdot & 1 \\ \phi_1 & \phi_0 & \cdot \\ \phi_2 & \phi_1 & \cdot \end{vmatrix} = (-1)^2 \begin{vmatrix} \phi_1 & \phi_0 \\ \phi_2 & \phi_1 \end{vmatrix}$$

$$F_2 = (-1)^2 \frac{1}{\phi_0^3} \begin{vmatrix} \phi_1 & \phi_0 \\ \phi_2 & \phi_1 \end{vmatrix}$$

For $F_3$, using four equations, we find

$$\Delta = \phi_0^4$$

$$\Delta F_3 = (-1)^3 \begin{vmatrix} \phi_1 & \phi_0 & \cdot \\ \phi_2 & \phi_1 & \phi_0 \\ \phi_3 & \phi_2 & \phi_1 \end{vmatrix}$$

$$F_3 = (-1)^3 \frac{1}{\phi_0^4} \begin{vmatrix} \phi_1 & \phi_0 & \cdot \\ \phi_2 & \phi_1 & \phi_0 \\ \phi_3 & \phi_2 & \phi_1 \end{vmatrix}$$

In general,

$$F_k = (-1)^k \frac{1}{\phi_0^{k+1}} \begin{vmatrix} \phi_1 & \phi_0 & \cdot & \cdot & \cdots & \cdot & \cdot \\ \phi_2 & \phi_1 & \phi_0 & \cdot & \cdots & \cdot & \cdot \\ \phi_3 & \phi_2 & \phi_1 & \phi_0 & \cdots & \cdot & \cdot \\ \phi_4 & \phi_3 & \phi_2 & \phi_1 & \cdots & \cdot & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \phi_{k-1} & \phi_{k-2} & \phi_{k-3} & \phi_{k-4} & \cdots & \phi_1 & \phi_0 \\ \phi_k & \phi_{k-1} & \phi_{k-2} & \phi_{k-3} & \cdots & \phi_2 & \phi_1 \end{vmatrix}$$

$$= (-1)^k \frac{1}{\phi_0^{k+1}} D_k$$

which gives us

$$\frac{1}{\Phi} \equiv \sum_{k=0}^{\infty}(-1)^k \frac{1}{\phi_0^{k+1}} D_k$$

for since we used a power series in $x$ to determine the $F_k$, we have the theorem rigorously stated

$$\equiv \underset{x\to 1}{\mathrm{L}} \frac{1}{\Sigma \phi_k x^k} \equiv \underset{x\to 1}{\mathrm{L}} \sum F_k x^k = \sum F_k$$

(4) Special cases would modify the general determinant $D_k$.

(a) $\phi_0 = 1$; $\phi_1$ and $\phi_2$ only present; we have

$$F_k = (-1) \begin{vmatrix} \phi_1 & 1 & \cdot & \cdot & & \cdot \\ \phi_2 & \phi_1 & 1 & \cdot & & \cdot \\ \cdot & \phi_2 & \phi_1 & 1 & & \cdot \\ \cdot & \cdot & \phi_2 & \phi_1 & & \cdot \\ & & & & \phi_1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \phi_2 & \phi_1 \end{vmatrix}$$

(b) $\phi_0 = 1$, $\phi_1$, $\phi_3$ present, $\phi_2 = 0$; we have

$$F_k = (-1)^k \begin{vmatrix} \phi_1 & 1 & \cdot & \cdot & & \cdot \\ \cdot & \phi_1 & 1 & \cdot & & \cdot \\ \phi_3 & \cdot & \phi_1 & 1 & & \cdot \\ \cdot & \phi_3 & \cdot & \phi_1 & & \cdot \\ & & & & \phi_1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \phi_1 \end{vmatrix}$$

etc.

II. *The Operators F.* (1) Since the $\phi_j$ are operators, the $F_k$ are also operators. The $\phi_j$ are of the form

$$\phi_j \equiv f_j(\vartheta) \cdot e^{k_j z}$$

If we operate on the left on the $(z, \vartheta)$ form by $[f_0(\vartheta)]^{-1}$ the $\phi_j$ take the form

$$\phi_j \equiv \frac{f_j(\vartheta)}{f_0(\vartheta)} e^{k_j z}, \qquad \text{and} \qquad \phi_0 = 1$$

Let us also set the $k_j = 1, 2, 3, \cdots$ The $F_k$ will then be, respectively,

$$F_1 = -\frac{f_1(\vartheta)}{f_0(\vartheta)} e^z$$

$$F_2 = (-1)^2 \begin{vmatrix} \frac{f_1(\vartheta)}{f_0(\vartheta)}e^z & 1 \\ \frac{f_2(\vartheta)}{f_0(\vartheta)}e^{2z} & \frac{f_1(\vartheta)}{f_0(\vartheta)}e^z \end{vmatrix}$$

$$F_3 = (-1)^3 \begin{vmatrix} \frac{f_1(\vartheta)}{f_0(\vartheta)}e^z & 1 & \cdot \\ \frac{f_2(\vartheta)}{f_0(\vartheta)}e^{2z} & \frac{f_1(\vartheta)}{f_0(\vartheta)}e^z & 1 \\ \frac{f_3(\vartheta)}{f_0(\vartheta)}e^{3z} & \frac{f_2(\vartheta)}{f_0(\vartheta)}e^{2z} & \frac{f_1(\vartheta)}{f_0(\vartheta)}e^z \end{vmatrix}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$F_k = (-1)^k \cdot$$

$$\begin{vmatrix} \frac{f_1(\vartheta)}{f_0(\vartheta)}e^z & 1 & \cdot & \cdot & \cdot \\ \frac{f_2(\vartheta)}{f_0(\vartheta)}e^{2z} & \frac{f_1(\vartheta)}{f_0(\vartheta)}e^z & 1 & \cdot & \cdot \\ \frac{f_3(\vartheta)}{f_0(\vartheta)}e^{3z} & \frac{f_2(\vartheta)}{f_0(\vartheta)}e^{2z} & \frac{f_1(\vartheta)}{f_0(\vartheta)}e^z & \cdot & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{f_{k-1}(\vartheta)}{f_0(\vartheta)}e^{(k-1)z} & \frac{f_{k-2}(\vartheta)}{f_0(\vartheta)}e^{(k-2)z} & \frac{f_{k-3}(\vartheta)}{f_0(\vartheta)}e^{(k-3)z} & \frac{f_1(\vartheta)}{f_0(\vartheta)}e^z & 1 \\ \frac{f_k(\vartheta)}{f_0(\vartheta)}e^{kz} & \frac{f_{k-1}(\vartheta)}{f_0(\vartheta)}e^{(k-1)z} & \frac{f_{k-2}(\vartheta)}{f_0(\vartheta)}e^{(k-2)z} & \frac{f_2(\vartheta)}{f_0(\vartheta)}e^{2z} & \frac{f_1(\vartheta)}{f_0(\vartheta)}e^z \end{vmatrix}$$

(2) We should like to carry the exponentials to the left across the $\vartheta$-operators by the shifting theorem $F(\vartheta)e^{nz} \equiv e^{nz}F(\vartheta + n)$ and find the resulting form of the determinant with the exponentials factored out to the left. By induction,

$$F_1 = (-1)\frac{f_1(\vartheta)}{f_0(\vartheta)}e^z = (-1)e^z\frac{f_1(\vartheta + 1)}{f_0(\vartheta + 1)}$$

$$F_2 = (-1)^2 \begin{vmatrix} \frac{f_1(\vartheta)}{f_0(\vartheta)}e^z & 1 \\ \frac{f_2(\vartheta)}{f_0(\vartheta)}e^{2z} & \frac{f_1(\vartheta)}{f_0(\vartheta)}e^z \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} e^z\frac{f_1(\vartheta + 1)}{f_0(\vartheta + 1)} & 1 \\ e^{2z}\frac{f_2(\vartheta + 2)}{f_0(\vartheta + 2)} & e^z\frac{f_1(\vartheta + 1)}{f_0(\vartheta + 1)} \end{vmatrix}$$

Since the form in position [2, 2], *i.e.*, in row 2 column 2, in the expansion is multiplied on the right of the form in position [1, 1],

its exponential will pass across that of [1, 1] and will have no effect on that in [2, 1]. Thus:

$$= (-1)^2 \begin{vmatrix} e^{2z}\dfrac{f_1(\vartheta+2)}{f_0(\vartheta+2)} & 1 \\ e^{2z}\dfrac{f_2(\vartheta+2)}{f_0(\vartheta+2)} & \dfrac{f_1(\vartheta+1)}{f_0(\vartheta+1)} \end{vmatrix}$$

And now we may factor $e^{2z}$ to the left:

$$= (-1)^2 e^{2z} \begin{vmatrix} \dfrac{f_1(\vartheta+2)}{f_0(\vartheta+2)} & 1 \\ \dfrac{f_2(\vartheta+2)}{f_0(\vartheta+2)} & \dfrac{f_1(\vartheta+1)}{f_0(\vartheta+1)} \end{vmatrix}$$

In $F_3$, we shall find also that the exponentials will affect the operator forms only diagonally upward to the left. In detail:

$$F_3 = (-1)^3 \begin{vmatrix} \dfrac{f_1(\vartheta)}{f_0(\vartheta)}e^z & 1 & \cdot \\ \dfrac{f_2(\vartheta)}{f_0(\vartheta)}e^{2z} & \dfrac{f_1(\vartheta)}{f_0(\vartheta)}e^z & 1 \\ \dfrac{f_3(\vartheta)}{f_0(\vartheta)}e^{3z} & \dfrac{f_2(\vartheta)}{f_0(\vartheta)}e^{2z} & \dfrac{f_1(\vartheta)}{f_0(\vartheta)}e^z \end{vmatrix}$$

Expand by elements of first row:

$$= (-1)^3 \left[ \frac{f_1(\vartheta)}{f_0(\vartheta)}e^z \begin{vmatrix} \dfrac{f_1(\vartheta)}{f_0(\vartheta)}e^z & 1 \\ \dfrac{f_2(\vartheta)}{f_0(\vartheta)}e^{2z} & \dfrac{f_1(\vartheta)}{f_0(\vartheta)}e^z \end{vmatrix} - \begin{vmatrix} \dfrac{f_2(\vartheta)}{f_0(\vartheta)}e^{2z} & 1 \\ \dfrac{f_3(\vartheta)}{f_0(\vartheta)}e^{3z} & \dfrac{f_1(\vartheta)}{f_0(\vartheta)}e^z \end{vmatrix} \right]$$

$$= (-1)^3 \left[ \frac{f_1(\vartheta)}{f_0(\vartheta)}e^{3z} \begin{vmatrix} \dfrac{f_1(\vartheta+2)}{f_0(\vartheta+2)} & 1 \\ \dfrac{f_2(\vartheta+2)}{f_0(\vartheta+2)} & \dfrac{f_1(\vartheta+1)}{f_0(\vartheta+1)} \end{vmatrix} - e^{3z} \begin{vmatrix} \dfrac{f_2(\vartheta+3)}{f_0(\vartheta+3)} & 1 \\ \dfrac{f_3(\vartheta+3)}{f_0(\vartheta+3)} & \dfrac{f_1(\vartheta+1)}{f_0(\vartheta+1)} \end{vmatrix} \right]$$

$$= (-1)\ e^{3z}\begin{vmatrix} \dfrac{f_1(\vartheta+3)}{f_0(\vartheta+3)} & \dfrac{f_1(\vartheta+2)}{f_0(\vartheta+2)} & 1 \\ & \dfrac{f_2(\vartheta+2)}{f_0(\vartheta+2)} & \dfrac{f_1(\vartheta+1)}{f_0(\vartheta+1)} \end{vmatrix}$$

$$- e^{3z}\begin{vmatrix} \dfrac{f_2(\vartheta+3)}{f_0(\vartheta+3)} & 1 \\ \dfrac{f_3(\vartheta+3)}{f_0(\vartheta+3)} & \dfrac{f_1(\vartheta+1)}{f_0(\vartheta+1)} \end{vmatrix}$$

$$(-1)^3e^{3z}\begin{vmatrix} \dfrac{f_1(\vartheta+3)}{f_0(\vartheta+3)} & 1 & \\ \dfrac{f_2(\vartheta+3)}{f_0(\vartheta+3)} & \dfrac{f_1(\vartheta+2)}{f_0(\vartheta+2)} & 1 \\ \dfrac{f_3(\vartheta+3)}{f_0(\vartheta+3)} & \dfrac{f_2(\vartheta+2)}{f_0(\vartheta+2)} & \dfrac{f_1(\vartheta+1)}{f_0(\vartheta+1)} \end{vmatrix}$$

(3) Now we may simplify still further by setting the units in the second diagonal column equal to the quotient of the respective denominators in that column by themselves. Thus:

$$1 = \frac{f_0(\vartheta+2)}{f_0(\vartheta+2)}, \qquad 1 = \frac{f_0(\vartheta+1)}{f_0(\vartheta+1)};$$

then factor out, so that

$$F_3 = (-1)^3e^{3z}\frac{1}{f_0(\vartheta+3)f_0(\vartheta+2)f_0(\vartheta+1)}\cdot\begin{vmatrix} f_1(\vartheta+3) & f_0(\vartheta+2) & \cdot \\ f_2(\vartheta+3) & f_1(\vartheta+2) & f_0(\vartheta+1) \\ f_3(\vartheta+3) & f_2(\vartheta+2) & f_1(\vartheta+1) \end{vmatrix}$$

(4) In like manner, we may easily show that, in general,

$$F_k = (-1)^ke^{kz}\frac{1}{\prod\limits_{h=1}^{k} f_0(\vartheta+h)}\begin{vmatrix} f_1(\vartheta+k) & f_0(\vartheta+k-1) & f_0(\vartheta+k-2) & \cdot & \cdot \\ f_2(\vartheta+k) & f_1(\vartheta+k-1) & f_1(\vartheta+k-2) & \cdot & \cdot \\ f_3(\vartheta+k) & f_2(\vartheta+k-1) & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_{k-1}(\vartheta+k) & f_{k-2}(\vartheta+k-1) & f_{k-3}(\vartheta+k-2) & f_1(\vartheta+2) & f_0(\vartheta+1) \\ f_k(\vartheta+k) & f_{k-1}(\vartheta+k-1) & f_{k-2}(\vartheta+k-2) & f_2(\vartheta+2) & f_1(\vartheta+1) \end{vmatrix}$$

where the subletters for the $f_i$ are the same down the diagonal columns, and the letters additive to $\vartheta$ are the same as, and down, the vertical columns from right to left.

III. *Max Mason's Theorem.** We shall quote this theorem in its entirety, with its proof.

(1) **If S is a convergent operator, the equation**

$$\mathbf{f = g + S \cdot f} \tag{a}$$

**admits a continuous solution, given by the equation**

$$\mathbf{f = g + S \cdot g + S^2 \cdot g + S^3 \cdot g +} \tag{b}$$

Consider the equation ($a$), where $g$ is a known continuous function for values of its arguments in a region $R$, and $S$ is a linear operator, so that $S(u + v) = Su + Sv$. It will be shown that a continuous solution may be found in a simple manner if the operator $S$ is of a certain type—is, namely, *convergent* according to the following definition. Let $\phi$ be any function continuous in the region $R$ considered: *The operator $S$ will* **be called convergent if the following conditions are satisfied:**

1° **$S\phi$ is a continuous function in R.**

2° **The infinite series $\phi + S\phi + S^2\phi + \cdots$ converges in R and represents a continuous function in R.**

3° **The result of the operation S on the function represented by this series is equal to the result of the term-by-term operation on the series.**

Suppose that $S$ is convergent and that a (continuous) solution $f$ of ($a$) exists. Then

$$\begin{aligned} f &= g + S \cdot f = g + S(g + Sf) = g + Sg + S^2f \\ &= g + Sg + S^2g + S^3f \end{aligned}$$

the equation

$$f = g + Sg + S^2g + \qquad + S^ng + S^{n+1}f$$

being obtained after $n$ substitutions of $g + Sf$ for $f$. If $S$ is convergent, the limit for $n = \infty$ may be taken. It follows that if a solution of ($a$) exists it has the form

$$f = g + Sg + S^2g + S^3g + \tag{b}$$

* Mason, Max, Selected Topics in the Theory of Boundary Problems of Differential Equations, *New Haven Math. Colloquium* (1910), 174. Mason states that Liouville probably used this method for the first time.

Conversely, if $S$ is a convergent operator the function defined by equation (*b*) is a solution of equation (*a*), for

$$Sf = Sg + S^2g + S^3g + \cdots = f - g.$$

(2) If we write (*a*) in the form

$$f - Sf = g$$

we have a differential equation with $S$ as a differential operator. If it is an inverse, say $\frac{1}{P}$, then

$$f - \frac{1}{P}f = g,$$

or

$$Pf - f = Pg;$$

*i.e.*, $(P - 1)f = Pg$, a differential equation. We might have written also

$$f - Sf \equiv (1 - S)f = g$$

or

$$f = \frac{1}{1 - S}g = \sum_{k=0}^{\infty} S^k \cdot g$$

We thus see a fundamental relation between an integral equation and a differential equation. The latter forms are fundamentally the forms that we are treating in this chapter under the section on Boole's theorem.

IV. *Boole's Theorem.** We shall quote this theorem also.

**The Linear Differential Equation**

$$\sum_{k=0}^{\overline{\ }} f_k(\vartheta) \cdot e^{kz} \cdot u = 0$$

**has as many [independent] solutions in ascending power series as there are simple [linear] factors in $f_0(\vartheta)$ and as many descending developments as there are factors of a like nature in $f_n(\vartheta)$.**

* BOOLE, GEORGE, A General Method in Analysis, *Phil. Trans., London* (1844), 225–282; see also Appendix III, §61.

(1) Operate on the left by $[f_0(\vartheta)]^{-1}$; *i.e.*,

$$\left[1 + \sum_{k=1}^{n} \frac{f_k(\vartheta)}{f_0(\vartheta)} e^{kz}\right] u = \frac{1}{f_0(\vartheta)} \cdot 0$$

The operation on the right depends on the factors of

$$f_0(\vartheta) \equiv \left|\begin{matrix} m \\ 1 \end{matrix}\right| (\vartheta - \alpha_{\underline{h}})$$

so that

$$\frac{1}{f_0(\vartheta)} \cdot 0 \equiv \frac{1}{\left|\begin{matrix} m \\ 1 \end{matrix}\right| (\vartheta - \alpha_{\underline{h}})} \cdot 0 = \sum_{h=1}^{m} C_h e^{\alpha_h z}$$

Then the operational solution will be

$$u = \frac{1}{1 + \sum_{k=1}^{n} \frac{f_k(\vartheta)}{f_0(\vartheta)}} \left[\sum_{h=1}^{m} C_h e^{\alpha_h z}\right]$$

(2) The necessary procedure now is to turn the inverse operator into a series of direct operators by operational division. When the summed term of the inverse is a single one, say

$$\frac{f_1(\vartheta)}{f_0(\vartheta)} e^z = S$$

then the direct operator series is a simple one:

$$\frac{1}{1 + S} \equiv 1 - S + S^2 - S^3 + - \cdots$$

we should have

$$S^k \equiv \left[\frac{f_1(\vartheta)}{f_0(\vartheta)} e^z\right]^k \equiv \frac{f_1(\vartheta)}{f_0(\vartheta)} e^z \cdot \frac{f_1(\vartheta)}{f_0(\vartheta)} e^z \cdots \frac{f_1(\vartheta)}{f_0(\vartheta)} e^z$$

When the summed term has more than one term, Boole merely suggested the possibility of obtaining the successive terms of the direct operator series by operational division, but he did not carry it out and did not apply it.

(3) Boole also operated to produce a descending power series; with

$$\left[\sum_{k=0}^{n} f_k(\vartheta)e^{kz}\right]y = 0$$

operate by $[f_n(\vartheta)]^{-1}$ on the left, obtaining

$$\left[\sum_{k=0}^{n-1} \frac{f_k(\vartheta)}{f_n(\vartheta)}e^{kz} + e^{nz}\right]y = [f_n(\vartheta)]^{-1}\cdot 0 = \sum_h C_h e^{\alpha_h z}$$

Then operate by $e^{-nz}$ on the left:

$$\left[e^{-nz}\sum_{k=0}^{n-1} \frac{f_k(\vartheta)}{f_n(\vartheta)}e^{kz} + 1\right]y = e^{-nz}\sum_h C_h e^{\alpha_h z}$$

or

$$\left[1 + \sum_{k=0}^{n-1} \frac{f_k(\vartheta + n)}{f_n(\vartheta + n)}e^{-(n-k)z}\right]y = \sum_h C_h e^{(\alpha_h - n)z}$$

This is essentially the same type of form as previously, except that it produces solutions in descending power series of the independent variable. Boole did not consider any inverse powers of intermediate forms obtained by dividing by $f_k(\vartheta)$ for $0 < k < n$, which would produce Laurent series solutions.

(4) Boole's solutions were also limited to those cases where the $f_0(\vartheta)$ and $f_n(\vartheta)$ had all unique factors, omitting cases of multiple linear factors or single or multiple irreducible quadratic factors. We shall remove all restrictions when we proceed with the solutions in general.

**§46. Bessel, Legendre, Hermite, and Hypergeometric Equations.**

(1) These equations illustrate the case for $i = 2$, which come exactly under Max Mason's theorem, *viz.*,

$$[f_0(\vartheta) + f_1(\vartheta)e^{k_1 z}]y = 0$$

or

$$\left[1 + \frac{f_1(\vartheta)}{f_0(\vartheta)}e^{k_1 z}\right]y = \frac{1}{f_0(\vartheta)}\cdot 0 = g$$

$$(1 + S)y = g$$

$$y = \frac{1}{1+S}\cdot g = \sum_{k=0}^{\infty} (-1)^k S^k \cdot g$$

or, in our form,

$$y = \sum_{k=0}^{\infty} F_k \cdot \left[\frac{1}{f_0(\vartheta)} \cdot 0\right]$$

(2) *The Bessel Equation of Order N.*

$$(x^2D^2 + xD + x^2 - n^2)y = 0, \qquad \text{the } (x, D) \text{ form}$$
$$[(\vartheta^2 - n^2) + e^{2z}]y = 0, \qquad \text{the } (z, \vartheta) \text{ form}$$
$$j = 2, \qquad i = 2, \qquad p = (2, 0), \qquad k_1 = 2$$

Symbol $[n, -n \mid 0 \mid\mid \cdot \mid 2]$

(3) Operate on the left by $(\vartheta^2 - n^2)^{-1}$, obtaining

$$\left[1 + \frac{1}{\vartheta^2 - n^2}e^{2z}\right]y = \frac{1}{\vartheta^2 - n^2} \cdot 0 = C_1e^{nz} + C_2e^{-nz}$$

Since there is but a single operator term, we have the direct operators in simple form, *i.e.*, with $S = \dfrac{1}{\vartheta^2 - n^2}e^{2z}$:

$$y = \sum_{k=0} (-1)^kS^k(C_1e^{nz} + C_2e^{-nz})$$

$$F_k \equiv (-1)^kS^k \equiv (-1)^ke^{k(2z)} \quad {}^{k} \quad \frac{1}{(\vartheta + 2\underline{h})^2 - n^2}$$

Operate on $C_1e^{nz}$ for the general term of the first-series solution:

$$F_kC_1e^{nz} = C_1(-1)^ke^{k(2z)} \quad {}^{k} \quad \overline{(\vartheta + 2\underline{h})^2 - n^2}$$

By substitution theorem,

$$= C_1(-1)^ke^{k(2z)}e^{nz} \quad {}^{k} \quad \frac{1}{(n + 2\underline{h})^2 - n^2}$$

(4) Operate on $C_2e^{-nz}$ for the general term of the second-series solution. If we use the direct substitution throughout, we shall find that the $n$th term of the series and all after it will have a factor whose denominator is zero. We shall, therefore, before operating, separate this term and use the substitution theorem on all others, but the shifting theorem and an integration on it. For $k < n$, we shall have a perfectly regular result in the form of a terminating series (polynomial); thus:

$$F_kC_2e^{-nz} = C_2e^{-nz}F_k \quad {\scriptstyle \vartheta=-} \qquad k < n$$
$$= C_2e^{-nz}(-1)e^{k(2z)}\left|\begin{matrix} k \\ 1 \end{matrix}\right|\frac{1}{(2\underline{h} - n)^2 - n^2},$$

which will give the polynomial

$$C_2e^{-nz}\left[1 + \sum_{k=1}^{n-1}(-1)^k e^{k(2z)}\left|\begin{matrix} k \\ 1 \end{matrix}\right|\frac{1}{(2\underline{h} - n)^2 - n^2}\right]$$

For $k \geqslant n$, we shall separate the operator, thus:

$$F_k = (-1)^k e^{k(2z)}\left[\frac{1}{(\vartheta + 2n)^2 - n^2}\right]\cdot\left|\begin{matrix} k \\ \neq n \\ 1 \end{matrix}\right|\frac{1}{(\vartheta + 2\underline{h})^2 - n^2}$$

Now operate on $C_2e^{-nz}$, thus:

$F_kC_2e^{-nz} =$

$$(-1)^k e^{k(2z)}\left[\frac{1}{(\vartheta + 2n)^2 - n^2}\right]\cdot\left|\begin{matrix} k \\ \neq n \\ 1 \end{matrix}\right|\frac{1}{(\vartheta + 2\underline{h})^2 - n^2}C_2e^{-nz} =$$

$$(-1)^k e^{k(2z)}\left[\frac{1}{(\vartheta + 2n)^2 - n^2}\right]\cdot C_2e^{-nz}\left|\begin{matrix} k \\ \neq n \\ 1 \end{matrix}\right|\frac{1}{(2\underline{h} - n)^2 - n^2},$$

[by the substitution theorem]

$$= (-1)^k e^{k(2z)}C_2e^{-nz}\frac{1}{(\vartheta - n + 2n)^2 - n^2}\cdot\left|\begin{matrix} k \\ \neq n \\ 1 \end{matrix}\right|\frac{1}{(2\underline{h} - n)^2 - n^2},$$

[by the shifting theorem]

Now, since

$$(\vartheta - n + 2n)^2 - n^2 = \vartheta(\vartheta + 2n)$$

and since $\left|\begin{matrix} k \\ \neq n \\ 1 \end{matrix}\right|\dfrac{1}{(2\underline{h} - n)^2 - n^2}$ is a constant, we have the last operation as

$$\frac{1}{\vartheta(\vartheta + 2n)}1 = \frac{1}{2n}z$$

giving us

$$F_kC_2e^{-nz} = C_2e^{-nz}(-1)^k e^{k(2z)}\frac{z}{2n}\left|\begin{matrix} k \\ \neq n \\ 1 \end{matrix}\right|\frac{1}{(2\underline{h} - n)^2 - n^2}, \qquad k \geqslant n$$

This now gives us the series for the second solution, *viz.*,

$$C_2 e^{-nz} \sum_{k=n} (-1)^k e^{k(2z)} \frac{z}{2n} \left| \begin{matrix} n \\ \neq n \\ 1 \end{matrix} \right| \frac{1}{(2\underline{h} - n)^2 - n^2}$$

(5) We may now combine the solutions and make the inverse transformation to variable $x$.

$$\begin{aligned} y = C_1 e^{nz} & \left[ 1 + \sum_{k=1}^{\infty} (-1)^k e^{k(2z)} \left| \begin{matrix} k \\ 1 \end{matrix} \right| \frac{1}{(2\underline{h} + n)^2 - n^2} \right] \\ & + C_2 e^{-nz} \left[ 1 + \sum_{k=1}^{n-1} (-1)^k e^{k(2z)} \left| \begin{matrix} k \\ 1 \end{matrix} \right| \frac{1}{(2\underline{h} - n)^2 - n^2} \right. \\ & \qquad \left. + \frac{z}{2n} \sum_{k=n}^{\infty} (-1)^k e^{k(2z)} \left| \begin{matrix} k \\ \neq n \\ 1 \end{matrix} \right| \frac{1}{(2\underline{h} - n)^2 - n^2} \right] \end{aligned}$$

or, in terms of $x$,

$$\begin{aligned} y = C_1 x^n & \left[ 1 + \sum_{k=1}^{\infty} (-1)^k x^{2k} \left| \begin{matrix} k \\ 1 \end{matrix} \right| \frac{1}{(2\underline{h} + n)^2 - n^2} \right] \\ & + C_2 x^{-n} \left[ 1 + \sum_{k=1}^{n-1} (-1)^k x^{2k} \left| \begin{matrix} k \\ 1 \end{matrix} \right| \frac{1}{(2\underline{h} - n)^2 - n^2} \right. \\ & \qquad \left. + \frac{1}{2n} \log x \cdot \sum_{k=n}^{\infty} (-1)^k x^{2k} \left| \begin{matrix} k \\ \neq n \\ 1 \end{matrix} \right| \frac{1}{(2\underline{h} - n)^2 - n^2} \right] \end{aligned}$$

This is the complete solution.

(6) *The Legendre Equation.*

$$[(1 - x^2)D^2 - 2xD + n(n + 1)]y = 0, \qquad \text{the } (x, D) \text{ form}$$

$$[\vartheta(\vartheta - 1) - (\vartheta - n - 2)(\vartheta + n - 1)e^{2z}]y = 0, \quad \text{the } (z, \vartheta) \text{ form}$$

$$j = 4, \qquad i = 2, \qquad p = (2, 2), \qquad k_1 = 2$$

$$\text{Symbol } [0, 1|\ 0\ ||\ n + 2, -n + 1\ |2]$$

(7) Operate on the left by $[\vartheta(\vartheta - 1)]^{-1}$:

$$\left[ 1 - \frac{(\vartheta - n - 2)(\vartheta + n - 1)}{\vartheta(\vartheta - 1)} e^{2z} \right] y = \frac{1}{\vartheta(\vartheta - 1)} \cdot 0 = C_1 + C_2 e^z$$

Here

$$F_k \equiv e^{k(2z)} \left| \begin{matrix} k \\ 1 \end{matrix} \right| \frac{(\vartheta - n - 2 + 2\underline{h})(\vartheta + n - 1 + 2\underline{h})}{(\vartheta + 2\underline{h})(\vartheta - 1 + 2\underline{h})}$$

(8) For the operation on $C_1$ we have

$$F_k \cdot C_1 \equiv C_1 F_k \cdot 1 = C_1 \cdot F_k \mid_{\vartheta=0}$$
$$= C_1 e^{k(2z)} \quad \frac{(2\underline{h} - n - 2)(2\underline{h} + n - 1)}{2\underline{h}(2\underline{h} - 1)}$$

Examining this for poles and zeros, we find

Poles: None, for $2h = 1$ gives $h = \frac{1}{2}$, and $h$ must be a positive integer

Zeros: $2h = n + 2, \qquad h = \frac{n}{2} + 1, \qquad k = \frac{n}{2}$, $n$ even and positive

$2h = -n + 1, \qquad h = \frac{-n+1}{2}, \qquad k = \frac{-n-1}{2}$, $n$ odd and negative

These values give terminating series (polynomials), called "Legendre polynomials." The series, therefore, will be

(a) ($n$ even and positive):

$$y_1 = C_1 \left[ 1 + \sum_{k=1}^{\frac{n}{2}} e^{k(2z)} \Big| \begin{matrix} k \\ 1 \end{matrix} \Big| \frac{(2\underline{h} - n - 2)(2\underline{h} + n - 1)}{2\underline{h}(2\underline{h} - 1)} \right]$$

(b) ($n$ odd and negative):

$$y_1 = C_1 \left[ 1 + \sum_{k=1}^{-\frac{n+1}{2}} e^{k(2z)} \Big| \begin{matrix} k \\ 1 \end{matrix} \Big| \frac{(2\underline{h} - n - 2)(2\underline{h} + n - 1)}{2\underline{h}(2\underline{h} - 1)} \right]$$

(c) ($n$ odd and positive):

$$y_1 = C_1 \left[ 1 + \sum_{k=1}^{\infty} e^{k(2z)} \Big| \begin{matrix} k \\ 1 \end{matrix} \Big| \frac{(2\underline{h} - n - 2)(2\underline{h} + n - 1)}{2\underline{h}(2\underline{h} - 1)} \right]$$

(d) ($n$ even and negative):

$$y_1 = C_1 \left[ 1 + \sum_{k=1}^{\infty} e^{k(2z)} \Big| \begin{matrix} k \\ 1 \end{matrix} \Big| \frac{(2\underline{h} - n - 2)(2\underline{h} + n - 1)}{2\underline{h}(2\underline{h} - 1)} \right]$$

(9) For the operation on $C_2e^z$, we have

$$F_kC_2e^z \equiv C_2 \cdot F_k e^z = C_2e^z \cdot F_k \Big|_{\vartheta=1}$$
$$= C_2e^z \cdot e^{k(2z)} \Big| \begin{matrix} k \\ 1 \end{matrix} \Big| \frac{(1 - n - 2 + 2\underline{h})(1 + n - 1 + 2\underline{h})}{(1 + 2\underline{h})(1 - 1 + 2\underline{h})}$$

Poles: None.

Zeros: $2h = n + 1, \quad h = \frac{n+1}{2}, \quad k = \frac{n-1}{2}, \quad n$ odd positive

$2h = -n, \quad h = -\frac{n}{2}, \quad k = -\frac{n+2}{2}, \quad n$ even negative

Here we have some more of the Legendre polynomials. The series for these will be

($a$) ($n$ even and positive):

$$y_2 = C_2e^z\left[1 + \sum_{k=1}^{\infty} e^{k(2z)} \Big| \begin{matrix} k \\ 1 \end{matrix} \Big| \frac{(2\underline{h} - n - 1)(2\underline{h} + n)}{(2\underline{h} + 1)2\underline{h}}\right]$$

($b$) ($n$ odd and negative):

$$y_2 = C_2e^z\left[1 + \sum_{k=1}^{\infty} e^{k(2z)} \Big| \begin{matrix} k \\ 1 \end{matrix} \Big| \frac{(2\underline{h} - n - 1)(2\underline{h} + n)}{(2\underline{h} + 1)2\underline{h}}\right]$$

($c$) ($n$ odd and positive):

$$y_2 = C_2e^z\left[1 + \sum_{k=1}^{\frac{n-1}{2}} e^{k(2z)} \Big| \begin{matrix} k \\ 1 \end{matrix} \Big| \frac{(2\underline{h} - n - 1)(2\underline{h} + n)}{(2\underline{h} + 1)2\underline{h}}\right]$$

($d$) ($n$ even and negative):

$$y_2 = C_2e^z\left[1 + \sum_{k=1}^{-\frac{n+2}{2}} e^{k(2z)} \Big| \begin{matrix} k \\ 1 \end{matrix} \Big| \frac{(2\underline{h} - n - 1)(2\underline{h} + n)}{(2\underline{h} + 1)2\underline{h}}\right]$$

(10) Using

$$A_k \equiv \Big| \begin{matrix} k \\ 1 \end{matrix} \Big| \frac{(2\underline{h} - n - 2)(2\underline{h} + n - 1)}{2\underline{h}(2\underline{h} - 1)}$$
$$B_k \equiv \Big| \begin{matrix} k \\ 1 \end{matrix} \Big| \frac{(2\underline{h} - n - 1)(2\underline{h} + n)}{(2\underline{h} + 1)2\underline{h}}$$

Combining and transforming to variable $x$, we have the complete solutions:

$$(a)\qquad y = C_1\left[1 + \sum_{k=1}^{\frac{n}{2}} x^{2k}A_k\right] + C_2x\left[1 + \sum_{k=1}^{\infty} x^{2k}B_k\right]$$

$$(b)\qquad y = C_1\left[1 + \sum_{k=1}^{-\frac{n+1}{2}} x^{2k}A_k\right] + C_2x\left[1 + \sum_{k=1}^{\infty} x^{2k}B_k\right]$$

$$(c)\qquad y = C_1\left[1 + \sum_{k=1}^{\infty} x^{2k}A_k\right] + C_2x\left[1 + \sum_{k=1}^{\frac{n-1}{2}} x^{2k}B_k\right]$$

$$(d)\qquad y = C_1\left[1 + \sum_{k=1}^{\infty} x^{2k}A_k\right] + C_2x\left[1 + \sum_{k=1}^{-\frac{n+2}{2}} x^{2k}B_k\right]$$

(11) It will be left to the student to solve the Hermite and hypergeometric equations for the ascending power series and to solve all four fqr the descending power series. We shall indicate the latter for the Hermite equation.

(12) *The Hermite Equation* (descending power-series solution):

$$(D^2 - xD + n)y = 0, \qquad (x, D) \text{ form}$$
$$[\vartheta(\vartheta - 1) - (\vartheta - n - 2)e^{2z}]y = 0, \qquad (z, \vartheta) \text{ form}$$
$$j = 3, \qquad i = 2, \qquad p = (2, 1), \qquad k_1 = 2$$
$$\text{Symbol } [0, 1|\ 0\ ||\ n + 2\ |2]$$

(13) Operate by $-(\vartheta - n - 2)^{-1}$:

$$[e^{2z} - (\vartheta - n - 2)^{-1}\vartheta(\vartheta - 1)]y = (\vartheta - n - 2)^{-1} \cdot 0 = Ce^{(n+2)z}$$

Now operate on the left by $e^{-2z}$:

$$\left[1 - e^{-2z}\frac{\vartheta(\vartheta - 1)}{\vartheta - n - 2}\right]y = Ce^{nz}$$

Now shift $e^{-2z}$ to the right over the $\vartheta$ form:

$$\left[1 - \frac{(\vartheta + 2)(\vartheta + 1)}{\vartheta - n}e^{-2z}\right]y = Ce^{nz}$$

Here $S$ is $\dfrac{(\vartheta + 2)(\vartheta + 1)}{\vartheta - n}e^{-2z}$, so that the operators $F_k$ will be in terms of $e^{-k(2z)}$, which will produce the descending power series desired. The student should carry out the work.

(14) Frequently there will appear multiple factors in $f_0(\vartheta)$ and $f_n(\vartheta)$, which will produce a somewhat different type of term for the $F_k$ to operate upon. We shall illustrate this by the Bessel equation of order zero.

$$(x^2D^2 + xD + x^2)y = 0, \qquad (x, D) \text{ form}$$
$$(\vartheta^2 + e^{2z})y = 0, \qquad (z, \vartheta) \text{ form}$$
$$j = 2, \qquad i = 2, \qquad p = (2, 0), \qquad k_1 = 2$$
$$\text{Symbol } [0, 0| \; 0 \,|| \cdot |2]$$

(15) Operate on the left by $\vartheta^{-2}$:

$$\left[1 + \frac{1}{\vartheta^2}e^{2z}\right]y = \frac{1}{\vartheta^2}\cdot 0 = C_1 + C_2z$$

The $F_k$ are here of the simple form

$$F_k \equiv (-1)^k e^{k(2z)}\left|\begin{matrix} k \\ 1 \end{matrix}\right|\frac{1}{(\vartheta + 2\underline{h})^2}$$

(16) The operation on $z$ is the interesting one, *viz.*,

$$\left|\begin{matrix} k \\ 1 \end{matrix}\right|\frac{1}{(\vartheta + 2\underline{h})^2}\cdot z$$

The continued product $\left|\begin{matrix} k \\ 1 \end{matrix}\right|\dfrac{1}{(\vartheta + 2\underline{h})^2}$ is expanded as a polynomial in $\vartheta$, and all terms in $\vartheta$ of higher degree than 1 are discarded because they produce zeros when operating on $z$. The product becomes the single term

$$1 - \sum_{h=1}^{k}\frac{1}{h}\vartheta$$

so that

$$\left|\begin{matrix} k \\ 1 \end{matrix}\right|\frac{1}{(\vartheta + 2\underline{h})^2} \equiv \left|\begin{matrix} k \\ 1 \end{matrix}\right|\frac{1}{(2\underline{h})^2}\cdot\frac{1}{\left(1 + \dfrac{\vartheta}{2\underline{h}}\right)^2}$$
$$= \left[\left|\begin{matrix} k \\ 1 \end{matrix}\right|\frac{1}{(2\underline{h})^2}\right]\cdot\left[1 - \sum_{h=1}^{k}\frac{1}{h}\vartheta\right]$$

Now operate on $z$:

$$\left[\left|\begin{matrix}k\\1\end{matrix}\right|\frac{1}{(2\underline{h})^2}\right]\cdot\left[1-\sum_{h=1}^{k}\frac{1}{h}\vartheta\right]z=\left|\begin{matrix}k\\1\end{matrix}\right|\frac{1}{(2\underline{h})^2}\left[z-\sum_{h=1}^{k}\frac{1}{h}\right]$$

which gives the form for the series; *i.e.*,

$$F_k\cdot z=(-1)^k e^{k(2z)}\left[z-\sum_{h=1}^{k}\frac{1}{h}\right]\cdot\left|\begin{matrix}k\\1\end{matrix}\right|\frac{1}{(2\underline{h})^2}$$

(17) The operation on 1 is

$$F_k\cdot 1=F_k|_{\vartheta=0}=(-1)^k e^{k(2z)}\left|\begin{matrix}k\\1\end{matrix}\right|\frac{1}{(2\underline{h})^2}$$

(18) We may now write down the complete solution

$$\begin{aligned}=C_1&\left[1+\sum_{k=1}^{\infty}(-1)^k x^{2k}\left|\begin{matrix}k\\1\end{matrix}\right|\frac{1}{(2\underline{h})^2}\right]\\&+C_2\left[\log x\left[1+\sum_{k=1}^{\infty}(-1)^k x^{2k}\left|\begin{matrix}k\\1\end{matrix}\right|\frac{1}{(2\underline{h})^2}\right]\right.\\&\qquad\left.-\left\{1+\sum_{k=1}^{\infty}(-1)^k x^{2k}\left[\sum_{h=1}^{k}\frac{1}{h}\right]\cdot\left|\begin{matrix}k\\1\end{matrix}\right|\frac{1}{(2\underline{h})^2}\right\}\right]\end{aligned}$$

(19) If there were a triple root, we should have $z^2$ to operate upon, in which case we could discard third powers and above in expanded product. Let the student verify the following operator forms: using

$$H_1=\sum_{h=1}^{k}\frac{1}{ha-\alpha},\qquad H_1G_1=\sum_{\substack{g,h=1\\g\neq h}}^{k}\frac{1}{ga-\alpha}\cdot\frac{1}{ha-\alpha}$$

$$H_2=\sum_{h=1}^{k}\frac{1}{(ha-\alpha)^2},\qquad H_1G_2=\sum_{\substack{g,h=1\\g\neq h}}^{k}\frac{1}{(ga-\alpha)^2}\cdot\frac{1}{ha-\alpha}$$

$$H_3=\sum_{h=1}^{k}\frac{1}{(ha-\alpha)^3},\qquad \text{etc.}$$

obtain

$$1 \Big| \frac{1}{(\vartheta - \alpha + a\underline{h})^3}$$

$$\equiv \left[ \left| \begin{matrix} k \\ 1 \end{matrix} \right| \frac{1}{(a\underline{h} - \alpha)^3} \right] \cdot [1 - 3H_1\vartheta + (6H_2 + 9G_1H_1)\vartheta^2 + \cdots]$$

$$1 \Big| \frac{1}{(\vartheta - \alpha + a\underline{h})^4}$$

$$\equiv \left[ \left| \begin{matrix} k \\ 1 \end{matrix} \right| \frac{1}{(a\underline{h} - \alpha)^4} \right] \cdot \left[ \begin{matrix} 1 - 4H_1\vartheta \\ +(10H_2 + 16G_1H_1)\vartheta^2 \\ -(20H_3 + 40G_1H_2)\vartheta^3 + - \cdots \end{matrix} \right]$$

$$\Big| \frac{1}{(\vartheta - \alpha + a\underline{h})^5}$$

$$\equiv \left[ \left| \begin{matrix} k \\ 1 \end{matrix} \right| \frac{1}{(a\underline{h} - \alpha)^5} \right] \cdot \left[ \begin{matrix} 1 - 5H_1\vartheta \\ +(15H_2 + 25G_1H_1)\vartheta^2 \\ -(35H_3 + 75G_1H_2 + 125F_1G_1H_1)\vartheta^3 \\ +\left[ \begin{matrix} 70H_4 + 175G_1H_3 + 225G_2H_2 \\ +625E_1F_1G_1H_1 + 375F_2G_1H_1 \end{matrix} \right]\vartheta^4 \\ - + \cdots \end{matrix} \right]$$

(20) After listing the following examples under $i = 2$, we shall proceed to the illustration of the solution for the general cases of $i = 3, 4, \cdots$

| $i$ | $j$ | $p$ | Equation | Name |
|---|---|---|---|---|
| 2 | 2 | (20) | $(x^2D^2 + xD + x^2 - n^2) \cdot y = 0$ | Bessel |
| | | | $[xD^2 + (\nu + 1)D + 1] \cdot y = 0$ | Bessel-Clifford |
| 2 | 3 | (21) | $(x^2D^2 + x^2D + kx + \frac{1}{4} - m^2) \cdot y = 0$ | Confluent hypergeometric |
| | | | $[xD^2 - (\alpha + 1 - x)D + n] \cdot y = 0$ | Laguerre |
| 2 | 4 | (22) | $(D^2 - xD + n) \cdot y = 0$ | Hermite |
| | | | $\{(x - x^2)D^2 + [\gamma - (\alpha + \beta + 1)x]D - \alpha\beta\} \cdot y = 0$ | Hypergeometric |
| | | | $[(1 - x^2)D^2 - 2xD + n(n + 1)] \cdot y = 0$ | Legendre |
| | | | $[(1 - x^2)D^2 - 3xD + n(n + 2)] \cdot y = 0$ | Tchebycheff |
| | | | $[(a + bx^n)x^2D^2 + (c + dx^n)xD + (f + gx^n)] \cdot y = 0$ | Pfaff |
| | | | $[(1 - x^2)D^2 - 2(\alpha + 1)xD + n(n + 2\alpha + 1)] \cdot y = 0$ | Ultraspherical |

**§47. The General Linear Equation.**

(1) We may now by similar methods attack the general equation

$$\sum_{m=0}^{j} f_m(\vartheta)e^{mz} \cdot y = 0$$

Operate on the left by $[f_0(\vartheta)]^{-1}$, and obtain

$$\left[1 + \sum_{m=1}^{j} \frac{{}_m(\vartheta)}{f_0(\vartheta)} e^{mz}\right] y = \frac{1}{f_0(\vartheta)} \cdot 0 = K$$

Here

$$\sum_{m=1}^{j} \frac{f_m(\vartheta)}{f_0(\vartheta)} e^{mz} \equiv \sum_{m=1}^{j} \phi_m$$

and the equation has the form

$$\left[1 + \sum_{m} \phi_m\right] y = K$$

which, when solved for $y$, gives

$$y = \frac{1}{1 + \sum_{m} \phi_m} \cdot K$$

(2) To be perfectly general, we must assume that every $m$ consecutively must be accounted for, and intermediate forms not present will be considered as being zeros. Thus, if we were to have

$$1 + \sum_{m} \phi_m \equiv 1 + \phi_2$$

we should write it as

$$\equiv 1 + 0 \cdot e^{z} + \frac{f_2(\vartheta)}{f_0(\vartheta)} e^{2z}$$

This will place the single $\phi_m$ in their proper places in the determinants making up the $F_k$. Thus, in this case,

$$F_k \equiv (-1)^k \begin{vmatrix} 0 & 1 & 0 & . & . & . & \\ \phi_2 & 0 & 1 & 0 & . & . & \\ 0 & \phi_2 & 0 & 1 & 0 & . & \\ . & 0 & \phi_2 & . & . & . & \\ . & . & 0 & . & . & . & . \\ . & . & . & . & . & . & . \\ & & & & . & . & . \end{vmatrix}$$

Only in this manner can the proper shifting of the exponentials be carried out.

(3) No additional remarks need be made as to the method in general, and we shall proceed to illustrate it by a well-known equation, *viz.*, the general Jacobi equation.

(4) *The Jacobi Equation:*

$$[(1 - x^2)D^2 - (\alpha + \beta + 2)xD + (\beta - \alpha)D + n(n + \alpha + \beta + 1)] \cdot y = 0, \qquad (x, D) \text{ form}$$

$$[\vartheta(\vartheta - 1) + (\beta - \alpha)(\vartheta - 1)e^z - (\vartheta + n + \alpha + \beta - 1)(\vartheta - n - 2)e^{2z}] \cdot y = 0, \qquad (z, \vartheta) \text{ form}$$

Here

$$i = 3, \qquad j = 5, \qquad p = (2, 1, 2),$$

$$\text{Symbol } [0, 1|\ 0\ ||\ 1\ |\ 1\ ||\ -n - \alpha - \beta + 1, n + 2\ |2]$$

(5) *The Functions:*

$$K \equiv \frac{1}{\vartheta(\vartheta - 1)} \cdot 0 = C_1 + C_2 e^z$$

$$\phi_1 \equiv \frac{(\beta - \alpha)(\vartheta - 1)}{\vartheta(\vartheta - 1)} e^z \equiv \frac{\beta - \alpha}{\vartheta} e^z$$

$$\phi_2 \equiv -\frac{(\vartheta + n + \alpha + \beta - 1)(\vartheta - n - 2)}{\vartheta(\vartheta - 1)} e^{2z}$$

(6) *The Operators:*

$$F_0 \equiv 1, \qquad F_1 \equiv \frac{\beta - \alpha}{\vartheta} e^z \equiv (\beta - \alpha) e^z \frac{1}{\vartheta + 1}$$

$$F_2 \equiv (-1)^2 \begin{vmatrix} \dfrac{(\beta-\alpha)}{\vartheta} e^z & 1 \\ -\dfrac{(\vartheta+n+\alpha+\beta-1)(\vartheta-n-2)}{\vartheta(\vartheta-1)} e^{2z} & \beta-\alpha \end{vmatrix}$$

$$\equiv (-1)^2 e^{2z} \begin{vmatrix} \dfrac{}{\vartheta+2} & 1 \\ -\dfrac{(\vartheta+2+n+\alpha+\beta-1)\cdot(\vartheta+2-n-2)}{(\vartheta+2)(\vartheta+2-1)} & \dfrac{\beta-\alpha}{\vartheta+1} \end{vmatrix}$$

$$F_3 \equiv (-1)^3 e^{3z} \begin{vmatrix} \dfrac{\beta-\alpha}{\vartheta+3} & 1 & 0 \\ -\dfrac{(\vartheta+3+n+\alpha+\beta-1)(\vartheta+3-n-2)}{(\vartheta+3)(\vartheta+3-1)} & \dfrac{\beta-\alpha}{\vartheta+2} & 1 \\ 0 & -\dfrac{(\vartheta+2+n+\alpha+\beta-1)(\vartheta+2-n-2)}{(\vartheta+2)(\vartheta+2-1)} & \dfrac{\beta-\alpha}{\vartheta+1} \end{vmatrix}$$

$F_k \equiv (-1)^k e^{kz} \cdot D_k$, where

$$D_k \equiv \begin{vmatrix} \dfrac{}{\vartheta+k} & \cdots & 0 & . \\ -\dfrac{(\vartheta+k+n+\alpha+\beta-1)(\vartheta+k-n-2)}{(\vartheta+k)(\vartheta+k-1)} & \dfrac{\beta-\alpha}{\vartheta+k-1} & 1 & . \\ 0 & -\dfrac{(\vartheta+k-1+n+\alpha+\beta-1)(\vartheta+k-1-n-2)}{(\vartheta+k-1)(\vartheta+k-1-1)} & \cdots & \end{vmatrix}$$

(7) *The Operations.* The $F_k$ operate on 1 and $e^z$, the respective integrals in the $K$. We need use only the $F_k$ and by it obtain the general term of the two series solutions.

Thus:

$$F_k \cdot 1 = F_k \mid_{\vartheta=0}, \qquad F_k \cdot e^z = e^z \cdot F_k \mid_{\vartheta=1}$$

$$F_k \mid_{\vartheta=0} = (-1)^k e^{kz} \begin{vmatrix} \frac{\beta-\alpha}{k} & 1 & 0 & \cdot & \\ -\frac{(k+n+\alpha+\beta-1)(k-n-2)}{k(k-1)} & \frac{\beta-\alpha}{k-1} & 1 & \cdot & \\ 0 & -\frac{(k-1+n+\alpha+\beta-1)(k-1-n-2)}{(k-1)(k-1-1)} & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \end{vmatrix}$$

$$F_k \mid_{\vartheta=1} = (-1)^k e^z e^{kz} \begin{vmatrix} \frac{\beta-\alpha}{k+1} & 1 & 0 & \cdot & \\ -\frac{(k+1+n+\alpha+\beta-1)\cdot(k+1-n-2)}{(k+1)(k+1-1)} & \frac{\beta-\alpha}{k} & 1 & \cdot & \\ 0 & -\frac{(k+n+\alpha+\beta-1)(k-n-2)}{k(k-1)} & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \end{vmatrix}$$

Thus we have both series completely determined in every detail as functions of $z$. It remains but to transform back to functions of $x$. Then, depending on the values of $\alpha$, $\beta$, and $n$, we may have numerical values of the determinants. We shall not make a study of these, as this section is meant only to illustrate the method of solution.

(8) A descending power series may be found in similar manner by operating on the left first by

$$[(\vartheta + n + \alpha + \beta - 1)(\vartheta - n - 2)]^{-1}$$

and then by $-e^{-2z}$, giving the form

$$[1 - \phi_1 - \phi_2] \cdot y = M$$

where $\phi_1 \equiv -\dfrac{(\beta - \alpha)(\vartheta + 1)}{(\vartheta + n + \alpha + \beta + 1)(\vartheta - n)} e^{-z}$

$$\phi_2 \equiv -\frac{(\vartheta + 2)(\vartheta + 1)}{(\vartheta + n + \alpha + \beta + 1)(\vartheta - n)} e^{-2z}$$

$$M \equiv \frac{1}{(\vartheta + n + \alpha + \beta - 1)(\vartheta - n - 2)} \cdot 0$$

$$= C_1 e^{-(n+\alpha+\beta-1)z} + C_2 e^{(n+2)z}$$

(9) A third series (a Laurent) may be found also by operating on the left first by $[(\beta - \alpha)(\vartheta - 1)]^{-1}$ and then by $+e^{-z}$, obtaining the form

$$[1 + \phi_1 - \phi_2] \cdot y = N$$

where $\phi_1 \equiv \dfrac{\vartheta + 1}{\beta - \alpha} e^{-z}.$

$$\phi_2 \equiv -\frac{(\vartheta + n + \alpha + \beta)(\vartheta - n - 1)}{(\beta - \alpha)\vartheta} e^{z}.$$

$$N \equiv \frac{1}{\vartheta - 1} \cdot 0 = C_1 e^{z}.$$

(10) These latter forms will be left to the student as exercises. Below is given also a set of classified examples which may be solved by the method of this section and chapter. They are all taken from the literature on differential equations and theory of functions. In most cases, the source author's name is given. Notation:

$$y_1 \equiv \frac{dy}{dx}, \qquad y_2 \equiv \frac{d^2y}{dx^2}, \text{ etc.}$$

$i \equiv$ the $f_i$ number, $j \equiv$ the class number, $p \equiv$ the partitions

| | | Equation | Source |
|---|---|---|---|
| | (11) | $(1 + x)y_1 = my$ | Ince |
| | (20) | $xy_2 + y_1 + y = ax^3$ | Cohen |
| | | $xy_2 + y_1 - y = 0$ | Ince |
| | | $x^2y_2 + xy_1 + mxy = 0$ | Forsythe |
| | | $y_2 + e^{2x}y = 0$ | Forsythe |
| | | $xy_2 + y_1 + xy = 0$ | Cohen |
| | | $4x^2y_2 + (1 - x^2)y = 0$ | Whittaker and Watson |
| | | $x^2y_2 + xy_1 - (x^2 + 1)^2y = 0$ | Smithsonian Mathematical Tables |
| | | $x^4y_2 + 2x^3y_1 - y = 0$ | Piaggio |
| 10 | | $x^2y_2 + 2xy_1 + c^3x^3y = 0$ | Forsythe |
| 11 | | $4xy_2 + xy_1 + y = 0$ | Ince |
| 12 | | $4xy_2 + 2y_1 + y = 0$ | |
| 13 | | [illegible]$y_2 - c^2y = 0$ | Forsythe |
| 14 | | $xy_2 + y = 0$ | Cohen |
| 15 | | [illegible]$_2 - xy = 0$ | Cohen |
| 16 | | [illegible]$y_2 + 9xy = 0$ | |
| 17 | | $y_2 + xy = 0$ | Ince |
| 18 | | $xy_2 - y_1 + y = 0$ | Forsythe |
| 19 | | $y_2 + x^2y = 0$ | Piaggio |
| 20 | | $xy_2 - 4y_1 + k^2xy = 0$ | Forsythe |
| 21 | | [illegible]$^3y_2 - (a^2 + \frac{5}{6}x)y = 0$ | Forsythe |
| 22 | | $x^4y_2 + 2x^3y_1 - (a^2 + 2x^2)y = 0$ | Forsythe |
| 23 | | $2x^2y_2 - xy_1 + (1 - x^2)y = 0$ | Cohen |
| 24 | | $2x^2y_2 - xy_1 + (1 - 2x^2)y = 0$ | Cohen |
| 25 | | $y_2 - n^2x^{2q-2}y = 0$ | Forsythe |
| 26 | | $y_2 - bcx^my = 0$ | Piaggio |
| 27 | | $2x^3y_2 + ax^2y_1 + 2by = 0$ | Forsythe |
| 28 | | $x^4y_2 + ax^3y_1 + by = 0$ | Forsythe |
| 29 | | $xy_2 + ny_1 + \frac{1}{4}y = 0$ | Forsythe |
| 30 | | $xy_2 + (\nu + 1)y_1 + y = 0$ | Smithsonian Mathematical Tables |
| 31 | | $xy_2 + (\nu - 1)y_1 + y = 0$ | Smithsonian Mathematical Tables |
| 32 | | $xy_2 + \mu y_1 + \lambda y = 0$ | Ince |
| 33 | | $xy_2 + x^2(q - 2m)y_1 + [x^2(m^2 - mq) - 2]y = 0$ | Forsythe |
| 34 | | $xy_2 + 2(n + 1)xy_1 \pm x^2y = 0$ | Smithsonian Mathematical Tables |
| 35 | | $x^2y_2 + 2nxy_1 \pm m^2x^2y = 0$ | Forsythe |
| 36 | (20) | $xy_2 + (2n + 1)y_1 + xy = 0$ | Whittaker and Watson |
| 37 | | $xy_2 + ay_1 + bxy = 0$ | Ince |
| 38 | | $xy_2 - ny_1 + x^{2n+1}y = 0$ | Piaggio |
| 39 | | $x^2y_2 + 3xy_1 + [x^2 - n^2 + 1]y = 0$ | Ince |
| 40 | | $x^2y_2 - a^2x^2y_1 - p(p + 1)y = 0$ | Forsythe |
| 41 | | $x^2y_2 + [x^2 - n(n + 1)]y = 0$ | Smithsonian Mathematical Tables |
| 42 | | $x^2y_2 + 2xy_1 + [x^2\mu^2 - n(n + 1)]y = 0$ | Smithsonian Mathematical Tables |
| 43 | | $x^2y_2 + 4(x + a)y = 0$ | Forsythe |
| 44 | | $y_2 + [Bx + c]y = 0$ | Whittaker and Watson |
| 45 | | $x^2y_2 - (2bx + c)y = 0$ | Ince |
| 46 | | $x^2y_2 + 2xy_1 - [n^2x - 2]y = 0$ | Forsythe |
| 47 | | $x^2y_2 + nxy_1 + (b + cx^{2m})y = 0$ | Forsythe |
| 48 | | $x^2y_2 - [n^2x^2 - m(m - 1)]y = 0$ | |
| 49 | (30) | $z^6w_3 + 6z^5w_2 - w = 0$ | Ince |
| 50 | | $y_3 + q^3y - \frac{6}{x^2}y_1 = 0$ | Forsythe |
| 51 | | $9x^2y_3 + 27xy_2 + 8y_1 - y = 0$ | Piaggio |
| 52 | (21) | $x^2y_2 - (1 - 3x)y_1 + y = 0$ | Piaggio |
| 53 | | $xy_2 + (1 + x)y_1 + 2y = 0$ | Cohen |
| 54 | | $xy_2 + (1 + ax^2)y_1 + bxy = 0$ | Forsythe |
| 55 | | $x^2y_2 + (3x - 1)y_1 + y = 0$ | Forsythe |
| 56 | | $x^2y_2 + (1 + 2x)xy_1 - 4y = 0$ | Forsythe |
| 57 | | $x^2y_2 + (2x + 1)xy_1 + (x - \nu)^2y = 0$ | Bateman |
| 58 | | $x^2y_2 + (x + a)y_1 - by = 0$ | Forsythe |
| 59 | | $x^5y_2 + xy_1 - (5 - 2x^3)y = 0$ | Forsythe |
| 60 | | $x^3y_2 + x(4a - x)y_1 - (6a + \frac{5}{4}x)y = 0$ | Forsythe |
| 61 | | $z^2w_2 + z^2w_1 - 2w = 0$ | Ince |
| | | $zw_2 + 2(1 - z)w_1 - w = 0$ | Ince |

| | | | Equation | Sour |
|---|---|---|---|---|
| | | | $x^4y_2 + xy_1 + y = 0$ | Cohen |
| | | | $x^3y_2 + 2xy_1 - y = 0$ | Forsythe |
| | | | $xy_2 + 2(1 - x)y_1 - y$ | |
| | | | $y_2 - xy_1 - xy = 0$ | |
| | | | $y_2 + x^2y_1 + xy = 0$ | Cohen |
| | | | $y_2 - x^2y_1 + xy = 0$ | Cohen |
| | | | $y_2 - xy_1 + ny = 0$ | |
| 70 | | | $zu_2 + (z + 1 + m)u_1 + \left(n + 1 + \frac{m}{2}\right)u = 0$ | Whittaker and Watson |
| | | | $y_2 - xy_1 + my = 0$ | Forsythe |
| | 2 \| 3 | (21) | $\cdots - (x - \gamma)y_1 - \alpha y = 0$ | |
| | | | $xy_2 - (\alpha + 1 - x)y_1 + ny = 0$ | |
| | | | $zu_2 + 2\mu u_1 - 2z^2u + 2(\nu - \mu)u = 0$ | Whittaker and Watson |
| | | | $y_2 - xy_1 - \alpha y = 0$ | Ince |
| | | | $x^2y_2 + ay_1 + by = 0$ | Forsythe |
| | | | $x^2y_2 + (2 + x)y_1 - 4y = 0$ | Forsythe |
| | | | $z^2w_2 + [pz - z^2]w_1 - rsw = 0$ | Ince |
| | 2 \| 4 | (22) | $(x - x^2)y_2 + (1 - 5x)y_1 - 4y = 0$ | |
| | | | $(x - x^3)y_2 + (1 - 3x^2)y_1 - xy = 0$ | Forsythe |
| | | | $x(1 - x)y_2 + [1 - (a + b + 1)x]y_1 - aby =$ | Forsythe |
| 82 | | | $(x - x^2)y_2 + (1 - x)y_1 - y = 0$ | Piaggio |
| 83 | | | $x^2(1 + x)y_2 - (1 + 2x)(xy_1 - y) = 0$ | Forsythe |
| 84 | | | $4(x^4 - x^2)y_2 + 8x^3y_1 - y = 0$ | Piaggio |
| 85 | | | $x(1 - x)y_2 - (1 + 3x)y_1 - y = 0$ | Piaggio |
| 86 | | | $x(1 - x)y_2 - 3xy_1 - y = 0$ | |
| 87 | | | $4x(1 - x)y_2 - (6 - 8x)y_1 - y = 0$ | Cohen |
| 88 | | | $x(x - 1)y_2 + (x + 2)y_1 - 4y = 0$ | Cohen |
| 89 | | | $(1 - x^3)y_2 - x^2y + xy = 0$ | Cohen |
| 90 | | | $x(x - 1)y_2 + (x - 3)y_1 - 4y = 0$ | Cohen |
| 91 | | | $x^2y_2 + x(1 + x)y_1 + (2x - 1)y = 0$ | Cohen |
| 92 | | | $x(1 - x)y_2 - 3y_1 + 2y = 0$ | |
| 93 | | | $2x(1 - x)y_2 + y_1 + 4y = 0$ | Forsythe |
| 94 | | | $2x(1 - x)y_2 + (1 - x)y_1 + 3y = 0$ | |
| 95 | | | $9x(1 - x)y_2 + 3(1 - 2x)y_1 + 20y = 0$ | Forsythe |
| 96 | | | $x(2 + x^2)y_2 - y_1 - 6xy_1 = 0$ | |
| 97 | | | $9x(1 - x)y_2 - 12y_1 + 4y = 0$ | Piaggio |
| 98 | | | $(1 - z^2)u_2 - 2zu_1 + 6u = 0$ | Whittaker and Watson |
| 99 | | | $2(2 - x)x^2y_2 - (4 - x)xy_1 + (3 - x)y = ($ | Forsythe |
| 100 | | | $(1 - x^2)y_2 + 2xy_1 + y = 0$ | Piaggio |
| 101 | | | $(1 - x^2)y_2 + 2xy_1 + 4y = 0$ | Cohen |
| 102 | | | $y_2 + x^2y_1 + xy = x^3$ | Cohen |
| 103 | | | $9x(1 - x)y_2 - 12xy_1 + 4y = 0$ | Cohen |
| 104 | | | $(1 - z^2)w_2 + 2(n - 1)zw_1 + 2nw = 0$ | |
| 105 | | | $(1 - ax^2)y_2 - bxy_1 - cy = 0$ | Forsythe |
| 106 | | | $(1 - x^2)y_2 - xy_1 + a^2y = 0$ | Forsythe |
| 107 | 2 \| 4 | (22) | $(1 - x^2)y_2 - 2(\alpha + 1)xy_1 + n(n + 2\alpha + 1)y = 0$ | |
| 108 | | | $(1 - z^2)u_2 - (2a + 1)zu_1 + n(n + 2a)u = 0$ | Whittaker and Watson |
| 109 | | | $(1 - x^2)y_2 + 2(m - 1)xy_1 + (n - m + 1)(n + m)y = 0$ | Forsythe |
| 110 | | | $(1 - x^2)y_2 - 2(m + 1)xy_1 + (n + m + 1)(n - m)y = 0$ | Forsythe |
| 111 | | | $(x^2 - 1)y_2 + 2(n + 1)y_1 - m(m + 2n + 1)y = 0$ | Whittaker and Watson |
| 112 | | | $x(1 - x)y_2 + [\gamma - (\alpha + \beta + 1)x]y_1 - \alpha\beta y = 0$ | |
| 113 | | | $z(zD + a)(zD + b)u - (zD - \alpha)(zD - \beta)u = 0$ | Whittaker and Watson |
| 114 | | | $(1 - x^2)y_2 + 2xy_1 + n(n + 1)y = 0$ | |
| 115 | | | $(x^2 - 1)y_2 + (2\mu - 1)xy_1 - \nu(\nu + 2\mu)y = 0$ | |
| 116 | | | $(1 - x^2)y_2 - 3xy_1 + n(n + 2)y = 0$ | |
| 117 | | | $x(1 - x)y_2 + \frac{1}{2}(\alpha + \beta + 1)(1 - 2x)y_1 - \alpha\beta y = 0$ | Whittaker and Watson |
| 118 | | | $x(x + 1)y_2 + 3y_1 - 2y = 0$ | Cohen |
| 119 | | (31) | $x^2y^3 + 3xy_2 + (1 - x)y_1 - y = 0$ | |

| | $i$ | $j$ | $p$ | Equation | Source |
|---|---|---|---|---|---|
| 120 | 2 | 5 | (32) | $y_3 - x^2y_2 - [1 - (a+b)]xy_1 - aby = 0$ | Forsythe |
| 121 | | | | $x^3y_3 + (x+3)x^2y_2 + (5x - 30)xy_1 + (4x + 30)y = 0$ | Forsythe |
| 122 | | 6 | (33) | $x^3(1+x)y_3 - 2(1+2x)x^2y_2 + 2(2+5x)xy_1 - 4(1+3x)y = 0$ | Forsythe |
| 123 | | | | $x^3(1+x^2)y_3 - 2(1+2x^2)x^2y_2 + 2(2+5x^2)xy_1 - 4(1+3x^2)y = 0$ | Forsythe |
| 124 | | | | $(1-x)x^2y^3 + [(\theta+\epsilon+1) - (\alpha+\beta+\gamma+3)x]xy_2 + [\theta\epsilon - (\alpha\beta+\beta\gamma+\gamma\alpha+\alpha+\beta+\gamma+1)x]y_1 - \alpha\beta\gamma y = 0$ | Ince |
| 125 | | $2n$ | $(2n,0)$ | $x^my_{2m} - y = 0$ | Forsythe |
| 126 | 3 | 2 | (200) | $4y_2 + (4n + 2 - x^2)y = 0$ | Ince |
| 127 | | | | $z^2y_2 - (az^2 + 2bz + c)y = 0$ | Ince |
| 128 | | | | $zy_2 - (1 + az^2)y = 0$ | Forsythe |
| 129 | | | | $4w_2 + (8k - z^2)w = 0$ | Whittaker and Watson |
| 130 | | | | $4x^2y_2 + (1 - 4m^2)y - (x^2 - 4kx)y = 0$ | Ince |
| 131 | | 3 | (210) | $xy_2 + (4x^2+1)y_1 + 4x(x^2+1)y = 0$ | Forsythe |
| 132 | | | | $xy_2 + [m + n + (\alpha+\beta)x]y_1 + [m\beta + n\alpha + \alpha\beta x]y = 0$ | Forsythe |
| 133 | | 3 | (120) | $x^2y_2 + y_1 - (3 + 2x)y = 0$ | Forsythe |
| 134 | | 4 | (220) | $z(1-z)u_2 + \frac{1}{2}(1-2z)u_1 + (az+b)u = 0$ | Whittaker and Watson |
| 135 | | | | $(2+x^2)y_2 + xy_1 + (1+x)y = 0$ | Piaggio |
| 136 | | 5 | (212) | $(1-x)x^2y_2 + (5x-4)xy_1 + (6-9x)y = 0$ | Forsythe |
| 137 | | | | $(1-x^2)y_2 + 2[b + (a-1)x]y_1 + 2ay = 0$ | Piaggio |
| 138 | | | | $(1-x^2)y_2 - (\alpha+\beta+2)xy_1 + (\beta-\alpha)y_1 + (n+\alpha+\beta+1)y = 0$ | |
| 139 | | | | $(1-x^2)y_2 - (m+1)xy_1 + (m - 1 - 2a)y_1 + (n+m)y = 0$ | |
| 140 | 3 | 6 | (222) | $(x + x^2 + x^3)y_2 + 3x^2y_1 - 2y = 0$ | Piaggio |
| 141 | | | | $(1-x^2)y_2 - 2xy_1 + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0$ | |
| 142 | | | | $(z-a)^2w_2 + A(z-a)w_1 + Bw = 0$ | Bieberbach |
| 143 | | | | $(z^3-1)^2u_2 + (z^3-1)2z^2u_1 + 9\gamma\gamma'zu = 0$ | Whittaker and Watson |
| 144 | | | | $(1-z^2)^2u_2 + n(n+2)u = 0$ | Whittaker and Watson |
| 145 | | 6 | (330) | $z^2(z^3+6)w_3 + (z^3+12)(3zw_2 + 3w_1 - z^2w) = 0$ | Ince |
| 146 | 4 | 3 | (2100) | $zw_2 - (1+z)w_1 + 2(1-z)w = 0$ | Ince |
| 147 | | 3 | (1110) | $x^2y_2 - (1 - 2x + 2x^2)y_1 + (1-x)^2y = 0$ | Forsythe |
| 148 | | 4 | (0220) | $x^4(1-x^2)y_2 + 2x^3y_1 - (1-x^2)^3y = 0$ | Piaggio |
| 149 | | 6 | (2121) | $z(2-z^2)w_2 - (z^2 - 4z + 2)\cdot[(1-z)w_1 + w] = 0$ | Ince |

## §48. Systems of Linear Differential Equations.

(1) Consider the set of equations

$$\sum_{j=1}^{n} F_{ij}(x, D) \cdot y_i = f_i(x), \qquad (i = 1, 2, \cdots, n) \tag{A}$$

where each $F_{ij}$ is of the form $\sum_h \phi_k(x) \cdot D^k$ and $\phi_k(x)$ is of the form $\sum_h a_h x^h$. Using the $\vartheta$-transformation [see §43], *i.e.*, $x = e^z$ and $x^nD^n \equiv \prod_{0}^{n-1}(\vartheta - \underline{\alpha})$, we get

$F_{ij}(x, D)$ to be of the form $\sum_k \psi_k(\vartheta) \cdot e^{kz}$, or $\Phi_{ij}(z, \vartheta)$

Then $(A)$ becomes

$$\sum_{j=1}^{n} \Phi_{ij}(z, \vartheta) \cdot y_i = f_i(e^z) \tag{B}$$

Here the characteristic determinant is

$$| \Phi_{ij}(z, \vartheta) |$$

and the characteristic equation is

$$| \Phi_{ij}(z, \vartheta) | = 0 \tag{C}$$

(2) The general complementary function $V$ of system $(A)$ is determined by the equation

$$\Phi_{ij}(z, \vartheta) \mid \cdot V = 0 \tag{D}$$

This is a single general linear differential equation similar to those treated in §47 of Chap. XI. Its solution is of the form

$$V = \sum C_m g_m(x) \tag{E}$$

where the $g_m(x)$ are particular solutions of $(D)$. The number $m$ is the degree in $\vartheta$ of the $\vartheta$-coefficient in expanded form of the characteristic determinant used as a divisor [see §45, Theorem IV]. These particular solutions $g_m(x)$ may be finite polynomials or infinite series, either ascending or descending, or Laurent. Equation $(D)$ is the eliminant of the homogeneous case of system $(A)$ and is the parallel of Eq. $(B)$ of Theorem I of §23.

(3) Two restrictions are placed on the characteristic determinant $| \Phi_{ij}(z, \vartheta) |$; it may not be identically zero in $\vartheta$ or a function of $z$ alone. In these cases respectively the function $V$ is zero or indeterminate.

(4) According to the algebraic theory of equations we may now write

$$y_j = \phi_{ij}(z, \vartheta) \cdot V, \qquad j = 1, 2, \cdots n; \text{ for every } i. \tag{F}$$

These are the *complementary functions* for the $y_j$, provided there are no common factors in all the $\phi_{ij}(\vartheta)$.

(5) If the cofactors $\phi_{ij}(\vartheta)$ have a common factor, a device similar to that in §24 (8) to (12) will be used to obtain the proper number of arbitrary constants in the complementary functions.

(6) Also if the characteristic determinant is singular, *i.e.*, if it is of rank less than the order of the set, a reduced number of equations will be used and arbitrary values assigned to some unknowns as in §24 (15). Reference is made to §24 for the details of method.

(7) For the *particular integrals* we have but to use Cramer's rule upon the set ($A$); *i.e.*,

$$| \Phi_{ij}(z, \vartheta) | \cdot y_j = K_i, \qquad i = 1, 2, \cdots, n$$

where $K_i$ is the characteristic determinant with the $j$th column replaced by the column of $f_i(e^z)$. Then

$$y_j = | \Phi_{ij}(z, \vartheta) |^{-1} \cdot K_i, \qquad j = 1, 2, \cdots, n \tag{G}$$

operationally performed give the integrals desired.

(8) The operational management of the integrals ($G$) is best done by turning $| \Phi_{ij}(z, \vartheta) |^{-1}$ into a power-series operator by the Reciprocal Theorem I [see §45]. That series may be obtained by a slight modification of the work of solving ($D$) in paragraph (2) of this section.

(9) Adding the particular integrals to the complementary functions respectively will give the complete set of solutions.

(10) An illustrative example:

The system

$$\begin{aligned} x^3D^2 \cdot y_1 + x^2D \cdot y_2 &= 1 \qquad (x, D) \text{ form} \\ x^2D \cdot y_1 + x\,D \cdot y_2 &= 0 \end{aligned}$$

by the substitutions $x = e^z$, $xD \equiv \vartheta$, $x^2D^2 \equiv \vartheta(\vartheta - 1)$ becomes

$$\begin{aligned} \vartheta(\vartheta - 1) \cdot y_1 + \vartheta \cdot y_2 &= e^{-z} \qquad (z, \vartheta) \text{ form} \\ e^z\vartheta \cdot y_1 + \vartheta \cdot y_2 &= 0 \end{aligned}$$

Here

$$| \phi_{ij}(z, \vartheta) | \equiv \begin{vmatrix} \vartheta(\vartheta - 1) & \vartheta \\ e^z\vartheta & \vartheta \end{vmatrix} \equiv \vartheta(\vartheta - 1)\vartheta - e^z\vartheta \cdot \vartheta$$
$$\equiv (\vartheta - 1 - e^z)\vartheta^2$$

Then $| \Phi_{ij}(z, \vartheta) | \cdot V = 0$ is

$$(\vartheta - 1 - e^z)\vartheta^2 V = 0$$

Substitute $\vartheta^2 V = W$, and solve for $W$; thus,

$$\left(1 - \frac{1}{\vartheta - 1} e^z\right) W = \frac{1}{\vartheta - 1} \cdot 0 = C_1 e^z$$

In this

$$\phi \equiv \frac{1}{\vartheta - 1} e^z, \qquad \text{so that} \qquad \phi^k \equiv e^{kz} \qquad \frac{1}{\vartheta + \underline{h} - 1},$$

and

$$\phi^k \cdot e^z = e^{(k+1)z}\Big| \qquad \frac{1}{h} \equiv e^{(k+1)z} \frac{1}{k!},$$

whence

$$W = C_1 \sum_{k=0}^{\infty} \frac{1}{k!} e^{(k+1)}$$

Now by $(F)$ we have

$$\frac{y_1}{\vartheta} = \frac{y_2}{-e^z \vartheta} = V$$

Operate through by $\vartheta$

$$\frac{y_1}{1} = \vartheta \cdot \frac{y_2}{-e^z \vartheta} = \vartheta V$$

or

$$\frac{y_1}{1} = \frac{y_2}{-e^z} = \vartheta V$$

Then since $s + p = p \cdot k$ must be satisfied, we have, since $s = 1$, $p = 1$ and $k = 2$. Thus,

$$\frac{\vartheta y_1}{1} = \frac{\vartheta y_2}{-e^z} = \vartheta^2 V = W$$

Substitute $\mu_1 = \vartheta y_1$ and $\mu_2 = \vartheta y_2$, so that

$$\frac{\mu_1}{1} = \frac{\mu_2}{-e^z} = W, \ i.e.,$$

$$\mu_1 = W$$

$$\mu_2 = -e^z W.$$

Inversely

$$y_1 = \frac{1}{\vartheta} \cdot W + C_1 \qquad \text{and} \qquad y_2 = \frac{1}{\vartheta}(-e^z W) + C_3.$$

Since

$$\frac{1}{\vartheta}W = \frac{1}{\vartheta}C_1\sum_{k=0}^{\infty}\frac{1}{k!}e^{(k+1)z} = C_1\sum_{k=0}^{\infty}\frac{1}{(k+1)!}e^{(k+1)z}$$

and

$$\frac{1}{\vartheta}(-e^zW) = -\frac{1}{\vartheta}C_1\sum_{k=0}^{\infty}\frac{1}{k!}e^{(k+2)z} = -C_1\sum_{k=0}^{\infty}\frac{1}{(k+2)\cdot k!}e^{(k+2)z}$$

we have the respective complementary functions for $y_1$ and $y_2$.

The particular integrals $\bar{y}_1$ and $\bar{y}_2$ are worked out as follows:

$$\bar{y}_1 = \frac{1}{\begin{vmatrix} \vartheta(\vartheta-1) & \vartheta \\ e^z\vartheta & \vartheta \end{vmatrix}}\cdot\begin{vmatrix} e^{-z} & \vartheta \\ 0 & \vartheta \end{vmatrix} = \frac{1}{\begin{vmatrix} \vartheta & \vartheta(\vartheta-1) \\ \vartheta & (\vartheta-1)e^z \end{vmatrix}}\cdot\begin{vmatrix} \vartheta & e^{-z} \\ \vartheta & 0 \end{vmatrix}$$

$$= \frac{1}{\vartheta(\vartheta-1)(\vartheta-e^z)}(-\vartheta e^{-z}) = \frac{1}{\vartheta-e^z}\cdot\frac{1}{\vartheta-1}\cdot\frac{1}{\vartheta}\cdot\vartheta(-e^{-z})$$

$$= \frac{1}{2e^z}\cdot(1-\vartheta e^{-z})^{-1}e^{-z} \qquad [\text{since } \vartheta - e^z = (\vartheta e^{-z}-1)e^z]$$

$$= \frac{1}{2e^z}\sum_{k=0}^{\infty}(-1)^k e^{-(k+1)z}\begin{vmatrix} k \\ 1 \end{vmatrix}(\underline{h}+1)$$

$$\left[\begin{aligned} \text{since } \phi &\equiv \vartheta e^{-z} \\ \phi^k &\equiv e^{-kz}\begin{vmatrix} k \\ 1 \end{vmatrix}(\vartheta-\underline{h}) \\ \phi^k e^{-z} &= (-1)^k e^{-(k+1)z}\begin{vmatrix} k \\ 1 \end{vmatrix}(\underline{h}+1) \end{aligned}\right]$$

$$= \tfrac{1}{2}\sum_{k=0}^{\infty}(-1)^k e^{-(k+2)z}\cdot k!$$

$$\bar{y}_2 = \frac{-1}{\vartheta(\vartheta-1)(\vartheta-e^z)}\cdot\begin{vmatrix} \vartheta(\vartheta-1) & e^{-z} \\ e^z\vartheta & 0 \end{vmatrix}$$

$$= \frac{1}{\vartheta-e^z}\cdot\frac{1}{\vartheta-1}\cdot\frac{1}{\vartheta}\cdot e^z\vartheta e^{-z}$$

$$= \frac{1}{\vartheta-e^z}\cdot\frac{1}{\vartheta}1 \qquad \left[\text{for } \frac{1}{\vartheta}e^z\vartheta e^{-z} = \frac{\vartheta-1}{\vartheta}1 \quad \text{and } \frac{1}{\vartheta-1}\cdot\frac{\vartheta-1}{\vartheta}1 = \frac{1}{\vartheta}1\right]$$

$$= \frac{1}{\vartheta-e^z}\cdot z = \frac{1}{e^z}(1-\vartheta e^{-z})^{-1}\cdot z$$

Now, we have

$$(1 - \vartheta e^{-z})^{-1} \equiv (1 - \phi)^{-1} \equiv \sum_{k=0}^{\infty} \phi^k$$

$$\phi \equiv \vartheta e^{-z}, \qquad \phi^k \equiv e^{-kz} \left| \begin{matrix} k \\ 1 \end{matrix} \right| (\vartheta - \underline{h})$$

We can expand the products $\left| \begin{matrix} k \\ 1 \end{matrix} \right| (\vartheta - \underline{h})$ into polynomials in $\vartheta$ and need only the constant term and that of the first power of $\vartheta$ in each one; thus,

$$\left| \begin{matrix} k \\ 1 \end{matrix} \right| (\vartheta - \underline{h}) = (-1)^k [M_k - N_k \vartheta + \cdots ]$$

where

$$M_k \equiv {}_kP_k \equiv k!$$

and

$$N_k \equiv \sum_{h=1}^{k} \frac{1}{h} {}_kP_k$$

Thus,

$$\frac{1}{\vartheta - e^z} \cdot z = \sum_{k=0}^{\infty} (-1)^k e^{-kz} [M_k - N_k \vartheta + \cdots ] z$$

$$= \left\| \begin{matrix} \infty \\ 0 \end{matrix} \right\| (-1)^k e^{-kz} [M_k z - N_k] = \bar{y}_2$$

where

$$\left\| \begin{matrix} \infty \\ 0 \end{matrix} \right\| \equiv \sum_{k=0}^{\infty},$$

Finally,

$$y_1 = C_1 \left\| \begin{matrix} \infty \\ 0 \end{matrix} \right\| \frac{1}{(k+1)!} e^{(k+1)z} + C_2 + \tfrac{1}{2} \left\| \begin{matrix} \infty \\ 0 \end{matrix} \right\| (-1)^k k! e^{-(k+2)z}$$

$$y_2 = -C_1 \left\| \begin{matrix} \infty \\ 0 \end{matrix} \right\| \frac{1}{(k+2) \cdot k!} e^{(k+2)z}$$

$$+ C_3 + \left\| \begin{matrix} \infty \\ 0 \end{matrix} \right\| (-1)^k e^{-kz} [M_k z - N_k]$$

or, in terms of $x$,

$$y_1 = C_1 \left\| \begin{matrix} \infty \\ 0 \end{matrix} \right\| \frac{1}{(k+1)!} x^{k+1} + C_2 + \tfrac{1}{2} \left\| \begin{matrix} \infty \\ 0 \end{matrix} \right\| (-1)^k k! x^{-(k+2)}$$

$$y_2 = -C_1 \left\| \begin{matrix} \infty \\ 0 \end{matrix} \right\| \frac{1}{(k+2)k!} x^{k+2}$$

$$+ C_3 + \left\| \begin{matrix} \infty \\ 0 \end{matrix} \right\| (-1)^k x^{-k} [M_k \log x - N_k]$$

## CHAPTER XII

# THE DIFFERENTIAL EQUATION IN MATHEMATICAL PHYSICS

### §49. The Nature of a Differential Equation.

(1) Differential equations in one sense are a mysterious invention. As they are written down from but one point of view, they presumably represent but one fact; but in reality they include vastly more than that one fact—even more than is suggested by their form or implied in the notation used. In general, a differential equation is the description in mathematical symbols of an observed physical fact or set of facts; but for it to be really useful we must add other facts and transform it by well-known mathematical processes. To say, for instance, that a body under the influence of attraction of another has a force acting on it seems quite simple or trivial, but to see in that statement the picture of the relation between time and the velocity and position of the body when it moves under that influence is not quite so simple. We can, however, write down the statement of fact in mathematical language and proceed confidently with our mathematical transformations to derive our time-velocity and time-position relations. Let us illustrate.

(2) The stated fact is that the force of gravity is equal to $gm$ and downward. Symbolically,

$$F = -gm \tag{a}$$

Now, a force can be represented in calculus symbolism by $\frac{mdv}{dt}$, where $v$ stands for velocity, and $t$ for time. Thus,

$$m\frac{dv}{dt} = -gm \tag{b}$$

By observation we know that a force on a free body produces motion. We shall have to assume that throughout the motion

the force will remain the same. The body is in what is called a "uniform" field; *i.e.*, the lines of force are straight and parallel.

(3) Now apply the process of integration, first dividing out the $m$:

$$v = -gt + C_1 \tag{c}$$

Integration brings in velocity and an arbitrary constant; *i.e.*, the $C_1$ is unknown unless something else is brought into our picture. To say anything about the arbitrary constant, we must begin to particularize. Up to this point we have been talking about *any* body in *any position* in the field of force $-gm$.

(4) Now we must think of a *particular body* in a *particular position* at a *particular time* with a *particular state* of *rest or motion*. We may say that it is at rest at beginning time

$$v = 0, \qquad t = 0 \tag{d}$$

Inserting these in Eq. ($c$), we have that

$$C_1 = 0$$

which gives

$$v = -gt \tag{e}$$

This equation now applies only to bodies initially at rest but starting to move in the direction the force is pulling at beginning time.

(5) A velocity is represented in calculus by $\frac{dy}{dt}$, where $y$ is distance. Thus:

$$v = \frac{dy}{dt} = -gt \tag{e'}$$

Now integrate again:

$$y = -\frac{g}{2}t^2 + C_2 \tag{f}$$

and obtain distance and another arbitrary constant which requires some more particularizing. We may now talk about, say, a body at the position of origin of coordinates when time begins; *i.e.*,

$$y = 0, \qquad t = 0 \tag{g}$$

Insert these in Eq. ($f$), and obtain that

$$C_2 = 0$$

or

$$y = -\tfrac{1}{2}gt^2 \tag{$f'$}$$

This equation connects position with time. All three equations ($b$), ($e$), and ($f'$) apply to the same body and describe the force acting on it, its velocity for any time as long as it is in the uniform field $-gm$, and its position in that field. Whereas the original Eq. ($b$) was of boundless application, Eq. ($f'$) has a very restricted application.

(6) The reason that the calculus gives the differentiation-integration relationship among acceleration, velocity, and distance is as follows. The first derivative is defined as a rate of change of one variable with respect to another in terms of which it is known. If distance is $y$ and is a known function of time, then its first derivative with respect to time is a rate of change of distance with respect to time, or what we also call velocity. Naturally, then, a velocity, if a function of time, can by the inverse process of integration bring distance into the picture. Also, if velocity is a known function of time, its first derivative is the rate of change of velocity with respect to time, or what we call acceleration, or applied force per unit mass. Again, if acceleration is a known function of time, it can by the inverse process of integration bring velocity into the picture. Thus we may again by a second inverse process bring in distance. These mathematical processes inherently, therefore, possess the power to connect acceleration, velocity, and time. When we set $F = m\dfrac{dv}{dt}$ we insert our knowledge of the mathematical and physical relation between force and velocity. When we set $v = \dfrac{dy}{dt}$ we insert our knowledge of the similar relation between velocity and distance, both relations being inherent in the structure of the mathematical symbolism used to describe our actual observations.

(7) As another illustration, consider a body of mass $m$ at a distance $x$ from the center of the radiating field of force of the earth of mass $M$. The differential equation

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2} \tag{h}$$

states that the force is proportional to the inverse square of the distance. Each side represents a force; it really says: action and reaction are equal. It could have been written

$$m\frac{d^2x}{dt^2} = -k\frac{mM}{x^2} \tag{h'}$$

where $k$ is the universal gravitation constant.

(8) Now particularize:

$$\begin{array}{lll} (a) & x = C & t = 0 \\ (b) & v = 0, & t = 0 \\ (c) & \dfrac{d^2x}{dt^2} = g, & x = a \end{array} \tag{i}$$

Out of ($h$) and the use of ($i$) we obtain by two integrations the time-position relation for the body $m$ with respect to $M$, with $\mu$ and the two arbitrary constants of integration evaluated—a very particular picture as compared with that represented by Eq. ($h$).

(9) The foregoing represent the "force" type of differential equation. Another type is that of equality of momenta. The "law of natural growth or extinction" is representative:

$$mv = m\frac{dy}{dt} = \pm kmy$$

or

$$\frac{dy}{dt} = \pm ky$$

(10) The differential equations of differential geometry describe geometric properties of analytic curves and surfaces. These are particularly applicable to engineering use, for they enable one to describe geometrically various mechanical properties of the materials of construction. Bending moment, shear, and deflection in beams are related in a mathematical manner, as are force, velocity, and distance in the preceding illustrations.

(11) Differential equations relating to stresses and strains and resulting distortions and waves give us powerful tools in the study of elasticity.

(12) Now let us look at the symbols that appear in differential equations—pure mathematical and dynamical.

I. *Mathematical symbols:*

$t$, time

$x$, $y$, $z$, $s$, lengths

$u$, $v$, $w$; $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, $\frac{ds}{dt}$, velocities, rate of change of distance with respect to time

$a$, $\alpha$, $\frac{du}{dt}$, $\frac{dv}{dt}$, $\frac{dw}{dt}$, $v\frac{dv}{dx}$, $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$, $\frac{d^2s}{dt^2}$, accelerations, rate of change of velocity with respect to time

$ma$, $mv\frac{dv}{dx}$, $\frac{md^2s}{dt^2}$, force (mass $\times$ acceleration)

II. *Dynamical units and dimensions:*

$m$, $M$, mass

$x$, $y$, $z$, $s$, $L$, length

$t$, $T$, time

$v$, $\frac{L}{T}$, velocity

$a$, $\frac{v}{T}$, $\frac{L}{T^2}$, acceleration

$F$, $H$, $Ma$, $M\frac{v}{T}$, $M\frac{L}{T^2}$, force

$W$, $FL$, $MaL$, $Mv^2$, $M\frac{L^2}{T^2}$, work, energy, couple

$P$, $\frac{W}{T}$, $M\frac{L^2}{T^3}$, power, work per unit of time

$S$, $\frac{F}{L^2}$, $\frac{ML}{T^2} \cdot \frac{1}{L^2}$, $\frac{M}{T^2L}$, stress, force per unit area

$I$, $ML^2$, inertia

$Mv$, $M\frac{L}{T}$, momentum, quantity of motion, force of motion

$Mv^2$, $FL$, $M\frac{L^2}{T^2}$, energy, work

$V$, $\Omega$, potential; $H = -\frac{\partial\Omega}{\partial L}$, $F = -\frac{\partial V}{\partial L}$

$$Fa,\ Ma^2,\ \frac{ML^4}{T^4},\ \text{couple}$$

Any one or more of I and II may be elements or terms of differential equations. In the next section, we shall show the make-up of some well-known differential equations of mathematical physics.

**§50. Some Differential Equations of Physical Problems.**

(1) *The Equation of Energy* $m\dfrac{dv}{dt} = Xm$. Both sides represent force, *i.e.*, an action and reaction equality. Multiply by $v$:

$$mv\frac{dv}{dt} = Xmv$$

Now, since $v\dfrac{dv}{dt} = \dfrac{d}{dt}\left[\dfrac{1}{2}mv^2\right]$ and $\dfrac{1}{2}mv^2 = E =$ energy, we have

$$\frac{d}{dt}\left[\frac{1}{2}mv^2\right] = \frac{dE}{dt} = mXv$$

and the left side represents rate of change of energy.

(2) *Force Equations.* *a.* Forced oscillations:

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \mu x = X$$

Multiply through by $m$, and separate the second derivative term:

$$m\frac{d^2x}{dt^2} = Xm - kmv - \mu mx$$

In this form, it is a statement of the equality of action and reaction. The left side is the reaction and the right the actions. The term $-kmv$ represents a resistance proportional to the velocity (momentum); the term $-\mu mx$, a force proportional to the distance, or a restoring force; while $Xm$ is the sum total of the applied forces to maintain the oscillations, which are periodic in character. If the equation is used without the $m$, it represents only the equality of forces per unit mass. From the dimensional standpoint, the terms $kmv$ and $\mu mx$ are not in form of dimension $Ma$; but since they are in the equation with an $Ma$, they must be thought of as having that dimension. Thus we may say that

$k$ has dimension $\frac{1}{T}$ and, by the same reasoning, $\mu$ has dimension $\frac{1}{T^2}$.

*b.* Vertical oscillations of a ship:

$$\rho V\frac{d^2x}{dt^2} = -g\rho Ax$$

Here $\rho$ = density, $V$ = volume, $g$ = gravity, $A$ = area of cross section of water line, $x$ = distance vertically of metacenter. Since $g$ is an acceleration and $\rho Ax$ a mass, and $\rho V$ also a mass, both sides represent a force.

*c.* Lateral oscillations of a particle on a stretched wire:

$$m\frac{d^2x}{dt^2} = -4P\frac{x}{l}$$

The right side is a restoring force, and the left the impressed force.

*d.* Constant propelling force with resistance:

$$m\frac{du}{dt} = mf - m\phi(u)$$

Left side is the reaction; right side, first term is the constant propelling force, and second term the resistance of the medium. This latter term may take various forms:

$$\phi(u) = ku,\ ku^2,\ ku^3,\ \frac{k}{u},\ \text{etc.}$$

*e.* Motion of a chain over an edge:

$$\mu x\frac{du}{dt} = g\mu x - T$$

Here $\mu x$ is a mass, $\frac{du}{dt}$ an acceleration; both the reaction. The term $g\mu x$ is a weight, and $T$ a tension; together, the actions.

*f.* Damped oscillations, single impulse:

$$m\frac{d^2x}{dt^2} = -\mu mx - mk\frac{dx}{dt}$$

All terms are forces: $m\frac{d^2x}{dt^2}$, the impulse; $m\mu x$, the restoring force; and $mk\frac{dx}{dt}$, the resisting or damping factor.

*g*. An electric circuit:

$$L\frac{di}{dt} + Ri + \frac{q}{C} = E$$

Here the electromotive force $E$ is equated to the reactions in the circuit. Of course, at all times they are in equilibrium. $L\frac{di}{dt}$ is an e.m.f. of self-induction due to a coil in the circuit; $Ri$ is a reaction due to the current and the resistance of the wire; and $\frac{q}{C}$ is the back e.m.f. due to the accumulated charge in a condenser.

*h*. Pendulum:

$$m\frac{d^2\theta}{dt^2} = -m\frac{g}{l}\sin\theta$$

Both sides are forces, *i.e.*, action of gravity, reaction of the bob.

(3) *Other Types*. *a*. D'Alembert's equation:

$$P = E\frac{du}{dx}$$

Here $P$ is a stress; $\frac{du}{dx}$ is a strain; and $E$ is the coefficient of compressional elasticity.

*b*. Torsional oscillations:

$$I\frac{d^2\theta}{dt^2} = -K\frac{\theta}{l}$$

Here $I$ is moment of inertia; and $K\frac{\theta}{l}$, a restoring couple; $K$ is the modulus of torsion.

*c*. Equation of angular momentum:

$$\frac{d}{dt}(I\omega) \quad N$$

As this stands, it is in the nature of a moment equation, as $N$ is the sum of all external forces with respect to a fixed axis of rotation. Multiply by $\omega$:

$$\omega\frac{d}{dt}(I\omega) = N\omega$$

$$I\omega\frac{d\omega}{dt} = N\omega$$

$$I\frac{d\theta}{dt}\,\frac{d^2\theta}{dt^2} = N\omega$$

This is in the form of ($b$) above. Or, it has the form

$$\frac{d}{dt}\left[\frac{1}{2}I\omega^2\right] = N\frac{d\theta}{dt} \qquad \text{the energy equation for rotation}$$

(4) *Geometrical Differential Equations.* We shall not illustrate these, as they are fully covered in elementary texts on the calculus and differential equations.

(5) In advanced texts on mechanics, elasticity, strength of materials, etc., many other types of differential equations are found of higher order than the second, with constant or variable coefficients. Enough has been said already to indicate that they all, when arising from physical problems, must show some equilibrium condition or state an identity of forces for all times.

# CHAPTER XIII

## INITIAL OR TERMINAL CONDITIONS

### §51. Introduction.

(1) In §49 (Chap. XII), we illustrated the make-up and use of the differential equation of a falling body and noted the necessity of particularizing in our picture to make our equation useful. The differential equation stated a fact true for a double infinity of physical configurations. The particulars necessary to restrict the results of the mathematical operations of integration to a specific configuration we call "initial" or "terminal conditions." Initial is used in the sense of "at the start" or when we begin counting time, and terminal in the sense of "at an end point" initially or finally or "at some critical position" where a particular relation is seen to be true. A wide variety of such particular situations is available in most physical setups, but the most commonly used is to set time zero and the other coordinates according to the picture at that time. Or, we may fix coordinates at initial values arbitrarily and count time from that configuration. In experimental work, a piece of apparatus is set up; a force or motion provided ready to be set into operation; and, when the operation starts, the counting of time is begun.

(2) Now, with the differential equation alone nothing can be done. It has to be integrated to be effective. We shall suppose the integration to be completed and the solution to be in hand, with its complete set of arbitrary constants and particular integral. The pure mathematician is usually content when this is done; but not so the applied mathematician, physicist, or engineer. The last of these is even inclined to belittle the "general solution" with its "arbitrary constants" and, like Oliver Heaviside, to wish to get his "particular solution" without ever having to think of the constants; and any device that tends to hide the very presence of them is welcomed with great glee,

despite the very obvious fact that the particularizing process is always present in some form or other.

(3) The process of using initial or terminal conditions is an easy one, and we shall now discuss it in detail.

§**52.** (1) In *dynamical systems*, we may generally say that we shall start time zero when everything is at rest. That is, all coordinates are zero. This includes velocities as well as coordinates of position. External forces which are fed into the system may or may not be zero at start. Thus, if $x_i$ are coordinates of position, $\frac{dx_i}{dt}$, those of velocities, we may set down

$$x_i\Big|_{t=0} = 0, \qquad \frac{dx_i}{dt}\Big|_{t=0} = 0, \qquad t = 0, \qquad a_i\Big|_{t=0} = a_{i0}$$

for our set of initial conditions.

(2) In *mechanical systems*, certain of the coordinates may be zero, and others may not be zero. Whatever the latter are, when the system is set in motion, they are called the "initial values." Thus,

$$x_i\Big|_{t=0} = x_{i0}, \qquad \frac{dx_i}{dt}\Big|_{t=0} = v_{i0}, \qquad a_i\Big|_{t=0} = a_{i0}, \qquad t = 0$$

(3) In *geometrical problems*, the geometrical picture will provide us with a finite or infinite number of possibilities for such special conditions.

(4) In general, as many sets of initial or particular conditions are needed as the number of arbitrary constants in the general solution, or, we may say, as the order of the differential equation. These sets of conditions, however, must be independent of one another.

(5) For a single equation with $m$ arbitrary constants, we must have $m$ sets of conditions. Say,

$$y = \sum_{k=m}^{1} C_k z_k$$

where the $z_k$ are of the form $e^{\alpha_k x}$ or $x^k e^{\alpha_k x}$; then we must have

$$\left.\begin{array}{l} y = y_j \\ x = x_j \end{array}\right\} \qquad j = 1, 2, \cdots, m$$

These sets are inserted into the solution. We then have

$$y_j = \sum_{k=m}^{1} C_k z_k \Big|_{x=x_j} \qquad j = 1, 2, \cdots, m$$

These are a nonhomogeneous set of algebraic equations in the $m$ constants $C_k$. If the $z_k|_{x=x_j}$ are independent of one another, this set of equations can be solved for values of the $C_k$ in terms of the $y_j$ and $x_j$ by Cramer's rule. Let us illustrate this process.

*a. A dynamical system* (single equation). An electrical circuit where self-induction and capacitance neutralize each other.

The differential equation:

$$\frac{d^2i}{dt^2} + \frac{1}{LC}i = 0$$

$$\left(D^2 + \frac{1}{LC}\right)i = 0, \qquad \text{operational form}$$

Solution:

$$i = C_1 \cos kt + C_2 \sin kt, \qquad k = \sqrt{\frac{1}{LC}}$$

Initial conditions:

(1) $\qquad i = 0, \qquad t = 0$

(2) $\qquad i = I_{\max}, \qquad t = t_m \qquad$ (to be determined)

Insert (1) in the solution, obtaining

$$0 = C_1$$

or

$$i = C_2 \sin kt$$

Then, for the maximum,

$$\frac{di}{dt} = kC_2 \cos kt = 0$$

$$\cos kt = 0$$

$$kt = \frac{\pi}{2}, \qquad t_m = \frac{\pi}{2k}$$

Now insert (2):

$$I_{\max} = C_2 \sin \frac{\pi}{2} = C_2$$

*i.e.*,

$$i = I_{\max} \sin kt$$

*b*. A *dynamical system* (simultaneous equations). The path of a corpuscle of mass $m$ and charge $e$ repelled from a negatively charged sheet of zinc illuminated with ultraviolet light, under a magnetic field $H$ parallel to the surface.

The differential equations:

$$m\frac{d^2x}{dt^2} = Xe - He\frac{dy}{dt}$$

$$m\frac{d^2y}{dt^2} = He\frac{dx}{dt}$$

$$m,\ X,\ e \text{ constants},\ \omega = \frac{He}{m}$$

Solutions:

$$x = m(C_{11} \cos \omega t + C_{12} \sin \omega t) + C_{13} + \frac{Xm^2}{H} \qquad (1)$$

$$y = m(C_{11} \sin \omega t - C_{12} \cos \omega t) + C_{14} + \frac{Xm^2}{H}t \qquad (2)$$

Differentiate:

$$Dx = m\omega(-C_{11} \sin \omega t + C_{12} \cos \omega t) \qquad (3)$$

$$Dy = m\omega(C_{11} \cos \omega t + C_{12} \sin \omega t) + \frac{Xm^2}{H} \qquad (4)$$

The conditions:

$$(a) \qquad x = y = 0 = t$$
$$(b) \qquad Dx = Dy = 0 = t$$

Inserting (*a*) and (*b*) in (1), (2), (3), and (4):

$$\begin{aligned}
mC_1 \qquad\qquad + C_{13} \qquad\qquad &= -\frac{Xm^2}{H} \\
-mC_{12} \qquad\qquad + C_{14} &= 0 \\
m\omega C_{12} \qquad\qquad &= 0 \\
m\omega C_{11} \qquad\qquad &= -\frac{Xm^2}{H}
\end{aligned}$$

By Cramer's rule

$$C_{11} = -\frac{Xe}{\omega}, \qquad C_{12} = 0, \qquad C_{13} = \frac{Xem}{} - \frac{Xm^2}{H}, \qquad C_{14} = 0$$

Particular solutions:

$$x = \frac{Xem}{\omega^2}(1 - \cos \omega t)$$

$$y = \frac{Xem}{\omega^2}(\omega t - \sin \omega t)$$

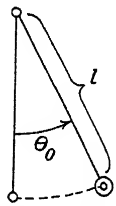
**§53. Geometrical Conditions.**

Conditions for the elimination of the arbitrary constants can often be obtained from the geometry of the physical situation. For instance, in §52 [illustration (*a*)], we obtained one of the sets of conditions from the zero slope of the $i = C_2 \sin kt$ curve, *i.e.*, for the maximum. Another such illustration would be the problem of the simple pendulum.

(*a*) A simple pendulum of length $l$ and mass $m$ swinging in a vacuum, amplitude small:

The differential equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$




Solution:

$$\theta = C_1 \cos kt + C_2 \sin kt, \qquad k = \sqrt{\frac{g}{l}}$$

Condition (1): $\theta = \theta_0$, $t = 0$, giving $\theta_0 = C_1 =$ maximum, or amplitude of swing

Condition (2):

$$\theta = 0, \qquad t = \tfrac{1}{4} \text{ period}$$

$$= \frac{1}{4}T = \frac{1}{4} \cdot \frac{2\pi}{k} = \frac{\pi}{2k}$$

$$0 = \theta_0 + C_2 \sin \frac{\pi}{2} = \theta_0 + C_2$$

$$C_2 = -\theta_0$$

*i.e.*,

$$\theta = \theta_0(\cos kt - \sin kt)$$

$$= \sqrt{2}\theta_0 \cos\left(kt - \frac{\pi}{2}\right)$$

The geometrical picture shows us that we could have obtained our first condition from either

$$\theta = \theta_0, \qquad t = 0, \qquad \text{or} \qquad \theta = 0, \qquad t = \tfrac{1}{4}T$$

and the second from either

$$\theta = -\theta_0, \qquad t = \tfrac{1}{2}T \qquad \text{or} \qquad \theta = 0, \qquad t = \tfrac{3}{4}T$$

Or, we could have started $t = 0$ when the bob was at the lowest point; thus:

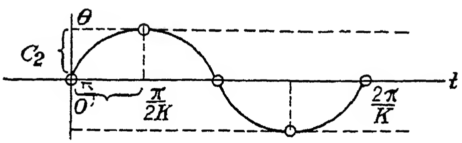
Condition (1):

$$\theta = 0, \qquad t = 0$$
$$0 = C_1$$

giving

$$\theta = C_2 \sin kt$$

Plotting,




Here

$$\theta = \theta_0, \qquad t = \frac{\pi}{2k}$$

or

$$\theta_0 = C_2 \sin \frac{\pi}{2} = C_2$$

giving

$$\theta = \theta_0 \sin kt$$

(*b*) Given the differential equation

$$(D^3 - 3D^2 + D - 3)y = \sin x \qquad \text{[Bateman]}$$

Determine the integral curve so that it may (1) have a point of inflexion at the origin and (2) have the $x$-axis as flexion tangent. The third condition, of course, is that the curve may pass through the origin.

Solution:

$$y = C_1e^{3x} + C_2 \cos x + C_3 \sin x + \frac{x}{8} \cos x$$

Conditions:

(3) $0 = C_1 + C_2$

(2) $y' = \frac{1}{8}(-x \sin x + \cos x) + 3C_1e^{3x} - C_2 \sin x + C_3 \cos x$

$0 = \frac{1}{8} + 3C_1 + C_3$

(1) $y'' = -\frac{1}{8}(x \cos x - 2 \sin x) + 9C_1e^{3x} - C_2 \cos x - C_3 \sin x$

$$0 = 9C_1 - C_2$$

Then

$$\begin{aligned} C_1 + C_2 \qquad\quad &= 0 \\ 3C_1 \qquad\quad + C_3 &= -\tfrac{1}{8} \\ 9C_1 - C_2 \qquad\quad &= 0 \end{aligned}$$

Cramer's rule gives

$$C_1 = C_2 = 0, \qquad C_3 = -\tfrac{1}{8}$$

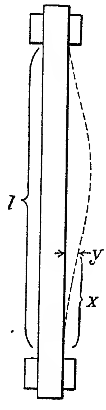
so that

$$y = \tfrac{1}{8}(x \cos x - \sin x)$$

(c) An interesting problem where one may use the same number of conditions as arbitrary constants, and still not obtain the latter uniquely is the following. A further condition is necessary.

The lateral displacement of a vertical shaft in rapid motion, supported in vertical bearings, is described by the differential equation

$$\frac{d^4y}{dx^4} - m^4y = 0$$




The solution is

$$y = C_1e^{mx} + C_2e^{-mx} + C_3e^{imx} + C_4e^{-imx}$$

$$y' = mC_1e^{mx} - mC_2e^{-mx} + imC_3e^{imx} - imC_4e^{-imx}$$

Since the deflexion $y$ is zero at both ends,

(1) $x = y = 0$

(2) $x = l, \qquad y = 0$

Also, since the shaft is constrained to be vertical at both ends,

$$(3) \qquad x = 0, \quad \frac{dy}{dx} = 0$$

$$(4) \qquad x = l, \quad \frac{dy}{dx} = 0$$

Inserting these conditions in the solution, we have

$$\begin{array}{lrrrrl}
(1) & C_1 + & C_2 + & C_3 + & C_4 & = 0 \\
(2) & e^{ml}C_1 + & e^{-ml}C_2 + & e^{iml}C_3 + & e^{-iml}C_4 & = 0 \\
(3) & mC_1 - & mC_2 + & imC_3 - & imC_4 & = 0 \\
(4) & me^{ml}C_1 - & me^{-ml}C_2 + & ime^{iml}C_3 - & ime^{-iml}C_4 & = 0
\end{array}$$

Dividing out the $m$ from (3) and (4), rearranging the equations, and writing out the determinant of the system, we have

$$\begin{vmatrix} 1 & -1 & i & -i \\ 1 & 1 & 1 & 1 \\ e^{ml} & e^{-ml} & e^{iml} & e^{-iml} \\ e^{ml} & -e^{-ml} & ie^{iml} & -ie^{-iml} \end{vmatrix} = \frac{e^{ml} + e^{-ml}}{2} \cdot \frac{e^{iml} + e^{-iml}}{2} - 1$$
$$= 0$$

as the condition for consistency. Now, since the rank of this determinant is 3, we may solve three of the equations for the proportions

$$\frac{C_1}{A_1} = \frac{C_2}{A_2} = \frac{C_3}{A_3} = \frac{C_4}{A_4}$$

where the $A_i$ are the cofactors of the elements of the first row. Thus, the four constants are not uniquely determined. We shall have to look at our picture again. Perhaps, but we are not sure, the maximum deflexion occurs midway, or at $x = \frac{l}{2}$, where $\frac{dy}{dx} = 0$. Let us try this.

$$y_{\max} = C_1e^{m\frac{l}{2}} + C_2e^{-m\frac{l}{2}} + C_3e^{im\frac{l}{2}} + C_4e^{-im\frac{l}{2}}$$

Inserting $C_1$, $C_2$, $C_3$ in terms of $C_4$:

$$y_{\max} = \frac{C_4}{A_4}\left[A_1e^{m\frac{l}{2}} + A_2e^{-m\frac{l}{2}} + A_3e^{im\frac{l}{2}} + A_4e^{-im\frac{l}{2}}\right]$$

or

$$C_4 = \frac{y_{\max}}{M} \qquad \text{where } M \text{ is the bracket.}$$

We see, therefore, that we shall have experimentally to determine the maximum deflexion and its location. As a matter of fact, we might determine any pair of values of $x$ and $y$ and find $C_4$. Thus, using $(x_1, y_1)$ as our experimental pair, we have

$$\begin{aligned} y_1 &= \frac{C_4}{A_4}[A_1e^{mx_1} + A_2e^{-mx_1} + A_3e^{imx_1} + A_4e^{-imx_1}] \\ &= \frac{C_4}{A_4}\cdot N \end{aligned}$$

or

$$C_4 = \frac{y_1A_4}{N}$$

Here, of course, neither $A_4$ nor $N$ may be zero. We then have as our particular solution, devoid of arbitrary constants,

$$y = \frac{y_1}{N}[A_1e^{mx} + A_2e^{-mx} + A_3e^{imx} + A_4e^{-imx}]$$

This needs also the preceding condition that

$$\cosh ml \cdot \cos ml = 1$$

$N$ is also the determinant of the system using $x_1$ in one row and then expanding by that row; *i.e.*,

$$N = \begin{matrix} e^{mx_1} & e^{-mx_1} & e^{imx_1} & e^{-imx_1} \\ 1 & 1 & 1 & 1 \\ e^{ml} & e^{-ml} & e^{iml} & e^{-iml} \\ e^{ml} & -e^{-ml} & ie^{iml} & -ie^{-iml} \end{matrix} \neq 0$$

**§54. Remarks.**

Of course solutions of differential equations are inexorably tied up with *existence theorems* and *regions of convergence* of any infinite series solutions. But a word of caution is necessary here, because most of the solvable problems of mathematical physics will give no trouble owing to the insertion of conditions in the solutions of the differential equations used for any finite picture. The differential equation usually covers exactly the

conditions in the region of experiment. The trouble comes when the physicist endeavors to reason mathematically without resort to experimental checking. He experiments mentally and obtains results that he asserts to be true without thoroughly checking *both* mathematically and physically throughout the range of his mental experimentation. The mathematical check must resort to finding the region of convergence of any infinite series in his solutions, and his physical check must involve just as wide a range of experimenting as is possible and then rejecting from his mathematics the region in which he cannot physically experiment. The scope of this book does not permit further discussion of these two questions, which will be found adequately covered in any good text on advanced function theory.

# APPENDIX I

## FORMULAS

### §55. Algebra.

(1) Distributive law: $m \cdot (a + b) \equiv m \cdot a + m \cdot b$

(2) Commutative law: $a \cdot b \equiv b \cdot a$

(3) Index law: $a^m \cdot a^n \equiv a^{m+n}$

(4) Partial fractions:

(a) $$\frac{f(x)}{\phi(x)} \equiv \sum \left[ \frac{A}{x-a}, \sum_k \frac{A_k}{(x-a)^k}, \frac{Ax+B}{(x-a)^2+b^2}, \sum_k \frac{A_k x + B_k}{[(x-a)^2+b^2]^k} \right]$$

(b) References:

EDWARDS, JOSEPH, "Integral Calculus," vol. I, §139, pp. 143*ff*.

WILLIAMSON, "Integral Calculus," §41, pp. 49*ff*.

RIETZ and CRATHORNE, "College Algebra," §124–127, pp. 177–181.

(5) Expansions into series of ascending or descending powers of the variable by algebraic division:

(a) $$\frac{f(x)}{\phi(x)} \equiv \sum_{n=0} a_n x^n$$

(b) $$\equiv \sum_{n=0} b_n x^{-n}$$

(6) Theory of Equations:

(a) Synthetic division.

(b) Horner's method.

(c) Newton's method.

(d) Reference: RIETZ and CRATHORNE, "College Algebra," Chap. XIII.

(7) Complex equality:

$$a + ib = x + iy, \qquad \text{only when} \qquad a = x, \qquad y = b$$

(8) Solution of a quadratic:

$$ax^2 + bd + c = 0$$

$$x = \frac{1}{2a}[-b \pm \sqrt{b^2 - 4ac}]$$

(9) Rationalization, by the use of factorization:

(a) $(x - a)(x + a) \equiv x^2 - a^2$

(b) $(x \pm a)(x^2 \mp ax + a^2) \equiv x^3 \pm a^3$

(c) $(x \pm a)\left[\sum_{k=0}^{n-1} (\mp 1)^k x^k a^{n-1-k}\right] \equiv x^n \pm a^n$

(10) Binomial theorem:

$$(a + b)^n \equiv \sum_{k=0}^{n} {}_nC_k \cdot a^{n-k} \cdot b^k$$

where ${}_nC_k \equiv \dfrac{n!}{(n - k)!\,k!}$

**§56. Trigonometry.**

(11) $\sin(\alpha \pm \beta) \equiv \sin\alpha \cdot \cos\beta \pm \cos\alpha \cdot \sin\beta$

(12) $\cos(\alpha \pm \beta) \equiv \cos\alpha \cdot \cos\beta \mp \sin\alpha \cdot \sin\beta$

(13) $a\cos x + b\sin x \equiv \sqrt{a^2 + b^2} \cdot \sin(x + \phi)$

where $\phi \equiv \arctan \dfrac{b}{a}$

(14) $\sin\alpha \cdot \cos\beta \equiv \tfrac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$

(15) $\cos\alpha \cdot \cos\beta \equiv \tfrac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$

(16) $\sin\alpha \cdot \sin\beta \equiv \tfrac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$

(17) $\sin \alpha x \equiv \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(\alpha x)^{2n+1}}{(2n + 1)!} \equiv \alpha x \cdot \prod_{n=0}^{\infty}\left(1 - \frac{\alpha^2 x^2}{n^2\pi^2}\right)$

$$\equiv \frac{1}{2i}(e^{i\alpha x} - e^{-i\alpha x}) \equiv -i \sinh i\alpha x$$

(18) $\sin \alpha x = 0, \qquad \alpha x = \pm k\pi, \qquad k \quad 0, 1, 2,$

(19) $\cos \alpha x \equiv \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(\alpha x)^{2n}}{(2n)!} \equiv \prod_{n=0}^{\infty}\left[1 - \frac{4\alpha^2 x^2}{(2n + 1)^2 \cdot \pi^2}\right]$

$$\equiv \tfrac{1}{2}(e^{i\alpha x} + e^{-i\alpha x}) \equiv \cosh i\alpha x$$

(20) $\cos \alpha x = 0, \qquad \alpha x = \pm \dfrac{2k + 1}{2}\pi, \qquad k = 0, 1, 2, \cdots$

(21) $\sinh \alpha x \equiv \sum_{n=0}^{\infty} \frac{(\alpha x)^{2n+1}}{(2n+1)!} \equiv \alpha x \cdot \prod_{n=1}^{\infty} \left[1 + \frac{\alpha^2 x^2}{n^2 \pi^2}\right]$

$\equiv \frac{1}{2}(e^{\alpha x} - e^{-\alpha x}) \equiv -i \sin i\alpha x$

(22) $\sinh \alpha x = 0, \qquad \alpha x = \pm ik\pi, \qquad k = 0, 1, 2, \cdot$

(23) $\cosh \alpha x \equiv \sum_{n=0}^{\infty} \frac{(\alpha x)^{2n}}{(2n)!} \equiv \prod_{n=0}^{\infty} \left[1 + \frac{4\alpha^2 x^2}{(2n+1)^2 \pi^2}\right]$

$\equiv \frac{1}{2}(e^{\alpha x} + e^{-\alpha x}) \equiv \cos i\alpha x$

(24) $\cosh \alpha x = 0, \qquad \alpha x = \pm i \cdot \frac{2k+1}{2} \cdot \pi,$

$k = 0, 1, 2,$

(25) $exp(\pm \alpha x) \equiv \sum_{n=0}^{\infty} \frac{(\pm \alpha x)^n}{n!} \equiv \cosh \alpha x \pm \sinh \alpha x$

(26) $exp(\pm i\alpha x) \equiv \sum_{n=0}^{\infty} \frac{(\pm i\alpha x)^n}{n!} \equiv \cos \alpha x \pm i \sin \alpha x$

(27) $(\cos \phi \pm i \sin \phi)^n \equiv \cos n\phi \pm i \sin n\phi$

[DeMoivre's Th.]

**57. Calculus.**

(28) $y = f(x); \qquad \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

(29) $z = f(x, y); \qquad dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

(30) $F(x, y, z) = 0; \qquad \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$

$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$

(31) Taylor's theorem: [Methodus Incrementorum (1715) p. 23]

(a) $f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a) \cdot (x - a)^k$

(b) $f(x + a) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a) \cdot x^k$ [Lagrange's notation]

(32) Maclaurin's theorem:

[James Stirling, "Lineae Tertii Ordinis Newtonianae," p. 32]

$a = 0, \qquad$ in (31b)

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0) \cdot x^k$$

(33) Leibnitz's theorem: [Commer. Epis. Leib. et Bern., II, 46, 99]

$$d^nW = \sum_{k=0}^{n} {}_nC_k \cdot d^ku \cdot d^{n-k}v, \qquad W = u \cdot v$$

(34) Taylor's theorem for two variables:

(a) $F(x, y)$

$$= \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \cdot F_{h,k}{}^{m+n}(a, b) \cdot (x - a)^m (y - b)^n$$

[Goursat-Hedrick, "Math. Anal.," II. I., p. 226 (12)]

(b) $F(x + h, y + k) =$

$$\sum_{m,n=0}^{\infty} \frac{\dfrac{\partial^{m+n}}{\partial x^m \cdot \partial y^n} \cdot F(x, y)}{m!n!} \cdot h^m \cdot k^n$$

[Goursat, *loc. cit.*, p. 224 (4)]

# APPENDIX II

## §58. Table of Operational Formulas.

With $Q \equiv p \cdot q \cdot r, \qquad Q^{-1} \equiv r^{-1} \cdot q^{-1} \cdot p^{-1}$ [I §3 (8)]

With $D \equiv \dfrac{d}{dx}, \qquad D^{-1} \equiv \int ()dx + c$

$$\equiv \int_a^x ()dx \qquad \text{[II §4 (1)]}$$

$$D^n \equiv D \cdot D \cdot D \cdots D$$

$$D^{-n} \equiv \frac{1}{D} \cdot \frac{1}{D} \cdot \frac{1}{D} \cdots \frac{1}{D} \equiv \int\int\int \cdots \int ()dx^n \equiv \sum_{i=0}^{n-1} C_i x^i$$

$$\equiv \int_a^x \int_a^x \cdots \int_a^x ()dx^n \tag{4}$$

$$D^m \cdot D^n \equiv D^{m+n} \tag{9}$$

$$D^m \cdot D^{-m} \equiv D^{-m} \cdot D^m \equiv 1 \tag{10}$$

$$[F(D)]^m \cdot [F(D)]^{-m} \equiv [F(D)]^{-m} \cdot [F(D)]^m \equiv 1$$

$$\frac{1}{D - a} \equiv \sum_{k=1}^{\infty} a^{k-1} D^{-k} \equiv -\sum_{k=0}^{\infty} a^{-k-1} D^k \tag{12}$$

Partial fraction types:

I. $(D - a)$ III. $[(D - a)^2 + b^2]$

II. $(D - a)^p$ IV. $[(D - a)^2 + b^2]^p$ (18)

Fundamental theorems in $D$: [§5 (1)]

I. $F(D) \cdot e^{\phi(x)} \equiv e^{\phi(x)} F[D + \phi'(x)]$

II. $F[x + \phi'(D)] \cdot e^{\phi(D)} \equiv e^{\phi(D)} \cdot F(x)$

III. $F(D^2) \left.\begin{matrix}\sin\\ \cos\end{matrix}\right] ax = F(-a^2) \left.\begin{matrix}\sin\\ \cos\end{matrix}\right] ax$

IV. $F(D^2) \left.\begin{matrix}\sinh\\ \cosh\end{matrix}\right] ax = F(a^2) \left.\begin{matrix}\sinh\\ \cosh\end{matrix}\right] ax$

V. $D^n \left[\prod_{i=1}^{m} u_i\right] \equiv \left[\sum_{i=1}^{m} D_i\right] \cdot \left[\prod_{k=1}^{m} u_k\right]$

$$\equiv \sum_{i=0}^{n} \frac{n!}{\prod_{k=1}^{m} (s_k!)} \cdot \prod_{k=1}^{m} D_k^{s_i} \cdot u_k$$

$$\text{VI.}\quad F(D) \equiv \sum_{k=0}^{\infty} \frac{1}{k!} D_1^k \cdot F^{(k)}(D_2)$$

$$\equiv e^{D_1 \cdot \frac{d}{dD_2}} F(D_2)$$

$$\text{I}a.\quad F^{-1}(D)e^{\phi(x)} \equiv e^{\phi(x)}F^{-1}[D + \phi'(x)] \qquad [\text{II} \quad \S 5 \quad (4)]$$

$$\text{I}b.\quad F(D)e^{ax} \equiv e^{ax}F(D + a) \qquad (5)$$

$$\text{I}c.\quad F(D)e^{ax}1 = e^{ax}F(a)$$

$$\text{I}d.\quad F^{-1}(D)e^{ax}1 = \frac{e^{ax}x^r}{F^{(r)}(a)} \qquad (6)$$

$$\text{II}a.\quad e^{\phi(D)}F^{-1}(x) \equiv F^{-1}[x + \phi'(D)]e^{\phi(D)} \qquad (8)$$

$$\text{II}b.\quad e^{kD}F(x) \equiv F(x + k)e^{kD} \qquad (9)$$

$$\text{II}c.\quad e^{kD}F(x)1 = F(x + k)$$

$$\text{II}d.\quad a^D F(x) \equiv F(x + \log_e a)a^D \qquad (10)$$

$$\text{II}e.\quad a^D F(x) \cdot 1 = F(x + \log_e a)$$

$$F(D) \equiv \phi(D^2) + D\psi(D^2) \qquad (12)$$

$$\text{V}a.\quad D^n(u \cdot v) \equiv \sum_{k=0}^{n} {}_nC_k \cdot D^k u \cdot D^{n-k} v \text{ [Leibnitz formula]} \qquad (17)$$

$${}_nC_k \equiv \frac{n!}{(n-k)!\,k!}$$

$$\text{V}b.\quad D^n(u \cdot v \cdot w)$$

$$\equiv (D_1 + D_2 + D_3)^n(u \cdot v \cdot w) \qquad (20)$$

$$\text{VII.}\quad \frac{1}{F} \cdot x^n \equiv \left[x - \frac{1}{F} \cdot F'\right]^n \cdot \frac{1}{F} \qquad (28)$$

$$(D - \alpha)^{-1} \equiv e^{\alpha x}D^{-1}e^{-\alpha x} \qquad [\S 6 \quad (3)]$$

$$\equiv \mu^{-1}e^{\alpha(x-u)} \equiv \int^x e^{\alpha(x-u)}()du \qquad (5)$$

$$(D - \alpha)^{-n}$$

$$\equiv e^{\alpha x}D^{-n}e^{-\alpha x} \qquad (6)$$

$$\equiv \mu^{-n}e^{\alpha(x-u)} \equiv \int^x \int^u \cdots \int^u e^{\alpha(x-u)}()du^n$$

$$\equiv \frac{1}{(n-1)!} \cdot \mu^{-1}(x - u)^{n-1}e^{\alpha(x-u)} \qquad (7)$$

$$\equiv \frac{1}{(n-1)!}\int^x (x - u)^{n-1}e^{\alpha(x-u)}()du$$

$$\prod^{n} (D - \alpha_i)^{-1} \equiv \sum_{i=1}^{n} A_i e^{\alpha_i x}D^{-1}e^{-\alpha_i x} \qquad (8)$$

$$\equiv \sum_{i=1}^{n} A_i \mu^{-1} e^{\alpha\,(x-u)}$$

$$\prod_{i=1}^{n} (D - \alpha_i)^{-k_i} \equiv \sum_{i=1}^{n} \left[ \sum_{s=1}^{k_i} N_{is}(D - \alpha_i)^{-s} \right] \qquad \text{[II} \quad \text{§6} \quad (9)]$$

$$\equiv \prod_{i=1}^{n} \mu^{-k_i} e^{\alpha_i(x-u)}$$

$$(D^2 + \beta^2)^{-1} \equiv \frac{1}{2\beta i}[(D - \beta i)^{-1} - (D + \beta i)^{-1}] \tag{10}$$

$$\equiv \frac{1}{\beta}\mu^{-1} \sin \beta(x - u)$$

$$\equiv \frac{1}{\beta}\int^{x} \sin \beta(x - u)\ ()du$$

$$[(D - \alpha)^2 + \beta^2]^{-1}$$

$$\equiv e^{\alpha x}(D^2 + \beta^2)^{-1} e^{-\alpha x} \tag{11}$$

$$\equiv \frac{1}{\beta}\mu^{-1} e^{\alpha(x-u)} \sin \beta(x - u)$$

$$\equiv \frac{1}{\beta}\int^{x} e^{\alpha(x-u)} \sin \beta(x - u)\ ()du$$

$$(D^2 + \beta^2)^{-n} \equiv \left[\frac{1}{\beta}\mu^{-1} \sin \beta(x - u)\right]^n \tag{12}$$

$$[(D - \alpha)^2 + \beta^2]^{-n} \equiv e^{\alpha x}(D^2 + \beta^2)^{-n} e^{-\alpha x} \tag{13}$$

$$D^{-n} \cdot 0 = \sum_{k=n-1}^{0} C_k x^k \qquad [\text{§7} \quad (2)]$$

$$(D - \alpha)^{-n} \cdot 0 = e^{\alpha x} D^{-n} \cdot 0 = e^{\alpha x} \sum_{k=n-1}^{0} C_k x^k \tag{3}$$

$$\prod_{i=1}^{m} (D - \alpha_i)^{-1} \cdot 0 = \sum_{i=1}^{m} C_i e^{\alpha_i x} \tag{4}$$

$$\prod_{i=1}^{m} (D - \alpha_i)^{-k_i} \cdot 0 = \sum_{i=1}^{m} \left[ e^{\alpha_i x} \sum_{s=0}^{k_i - 1} C_{is} x^s \right] \tag{5}$$

$$(D^2 + \beta^2)^{-1} \cdot 0 = K_1 e^{i\beta x} + K_2 e^{-i\beta x} \tag{6}$$

$$= C_1 \cos \beta x + C_2 \sin \beta x$$

$$[(D - \alpha)^2 + \beta^2]^{-1} \cdot 0$$

$$= e^{\alpha x}(D^2 + \beta^2)^{-1} \cdot 0$$

$$= e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x\} \tag{7}$$

$$(D^2 + \beta^2)^{-n} \cdot 0$$
$$= e^{-i\beta x} \sum_{h=0}^{n-1} C_{1h} x^h + e^{i\beta x} \sum_{h=0}^{n-1} C_{2h} x^h \qquad \text{[II §7 (8)]}$$
$$= \left[\sum_{s=0}^{n-1} K_{1s} x^s\right] \cos \beta x + \left[\sum_{s=0}^{n-1} K_{2s} x^s\right] \sin \beta x$$

$$[(D - \alpha)^2 + \beta^2]^{-n} \cdot 0 \equiv e^{\alpha x}(D^2 + \beta^2)^{-n} \cdot 0 \qquad (9)$$
[see II §7 (8)]

$$\prod_{i=1}^{m} (D^2 + \beta_i^2)^{-1} \cdot 0 = \sum_{i=1}^{m} (C_{1i} \cos \beta x + C_{2i} \sin \beta x) \qquad (10)$$

$$\prod_{k=1}^{m} [(D - \alpha_k)^2 + \beta_k^2]^{-1} \cdot 0$$
$$\equiv \prod_{k=1}^{m} e^{\alpha_k x}(D^2 + \beta_k^2)^{-1} \cdot 0 \qquad (11)$$
[see II §7 (10)]

$$\prod_{k=1}^{m} (D^2 + \beta_k^2)^{-h_k} \cdot 0 = \sum_{k=1}^{m} \left[\left(\sum_{s=0}^{h_k-1} A_{ks} x^s\right) \cos \beta_k x + \left(\sum_{s=0}^{h_k-1} B_{ks} x^s\right) \sin \beta_k x\right] \qquad (12)$$

$$\prod_{k=1}^{m} [(D - \alpha_k)^2 + \beta_k^2]^{-h_k} \cdot 0$$
$$= \prod_{k=1}^{m} e^{\alpha_k x}(D^2 + \beta_k^2)^{-h_k} \cdot 0 \qquad (13)$$
[see II §7 (12)]

$$D^{-n} \cdot 1 = \frac{x^m}{m!} \qquad \text{[§8 (2)]}$$
$$(D - \alpha)^{-n} \cdot 1 = (-\alpha)^{-n} \qquad (3)$$
$$\prod_{i=1}^{m} (D - \alpha_i)^{-1} \cdot 1 = \prod_{i=1}^{m} (-\alpha_i)^{-1} \qquad (4)$$
$$\prod_{i=1}^{m} (D - \alpha_i)^{-k_i} \cdot 1 = \prod_{i=1}^{m} (-\alpha_i)^{-k_i} \qquad (5)$$
$$(D^2 + \beta^2)^{-n} \cdot 1 = \beta^{-2n} \qquad (6)$$
$$[(D - \alpha)^2 + \beta^2]^{-n} \cdot 1 = [(-\alpha)^2 + \beta^2]^{-n} \qquad (7)$$

$$\prod_{i=1}^{m}[(D - \alpha_i)^2 + \beta_i^2]^{-1} \cdot 1 = \prod_{i=1}^{m}[(-\alpha_i)^2 + \beta_i^2]^{-1} \tag{8}$$

$$\prod_{i=1}[(D - \alpha_i)^2 + \beta_i^2]^{-k_i} \cdot 1 = \prod_{i=1}[(-\alpha_i)^2 + \beta_i^2]^{-k_i} \tag{9}$$

$$F(D) \cdot 1 = F(0), \qquad \text{provided} \qquad F(0) \neq \infty \tag{10}$$

$$F(D) \cdot 1 = \frac{1}{D^k G(D)} 1 = \frac{1}{G(0)} \cdot \frac{x^k}{k!} \tag{10}$$

$$F(D) \cdot y = f(x) \qquad \text{[III §11 (1)]}$$

$$y = F^{-1}(D) \cdot f(x) + F^{-1}(D) \cdot 0$$

Heaviside expansion theorem: [§12]

$$y \quad \frac{Y(0)}{Z(0)} + \sum_{p_1, p_2 \cdots} \frac{Y(p)e^{pt}}{p\frac{dZ}{dp}}$$

Systems of L.D.E. with C.C.: [VI]

$$\sum_{j=1}^{m} F_{ij}(D) \cdot y_j = X_i, \qquad i = 1, 2 \cdots m$$

Theorem I: $| F_{ij}(D) | \cdot V = 0$ [§23 (*B*)]

Characteristic equation: $| F_{ij}(D) | = 0$ (*C*)

Theorem II: $y_j = f_{ij} \cdot V$ (*D*)

$$s + p = p \cdot q \tag{10}$$

Theorem III: $y_j = | F_{ij}(D) |^{-1} \cdot | K_j |$ 25 (*M*)

$$d_i \equiv \frac{\partial}{\partial x_i}; \qquad \text{[VII §28 (1)]}$$

Special case of two variables:

$$d_1 \equiv \frac{\partial}{\partial x}, \qquad d_2 \equiv \frac{\partial}{\partial y}$$

$$d_1^{-1} \equiv \int^x ()\partial x + f(y) \equiv \int_a^x ()\partial x \tag{2}$$

$$d_2^{-1} \equiv \int^y ()\partial y + \phi(x) \equiv \int_b^y ()\partial y$$

$$d_1^{-n} \equiv \int^x \int^x \cdots \int^x ()\partial x^n + E(y)$$

$$\equiv \int_a^x \int_a^x \cdots \int_a^x ()\partial x^n \tag{3}$$

$$d_2^{-n} \equiv \int^y\int^y \cdots \int^y ()\partial y^n + F(x)$$

$$\equiv \int_b^y\int_b^y \cdots \int^y ()\partial y^n$$

$$d_1 d_2 \equiv d_2 d_1 \qquad \text{[VII §28 (5)]}$$

$$f(d_1, d_2) \cdot \phi(d_1, d_2) \equiv \phi(d_1, d_2) \cdot f(d_1, d_2)$$

but

$$f(d_1, d_2, x, y) \cdot \phi(d_1, d_2, x, y) \not\equiv \phi \cdot f$$

$$d_i^m d_i^n \equiv d_i^{m+n}$$

$$f^m \cdot f^n \equiv f^{m+n} \qquad (6)$$

$$d_i^m d_i^{-n} \equiv d_i^{-n} d_i^m$$

$$d_i^m d_i^{-m} \equiv d_i^{-m} d_i^m \equiv d_i^o \equiv 1 \qquad (7)$$

$$f^m f^{-m} \equiv f^{-m} f^m \equiv f^o \equiv 1$$

Partial fraction forms in two variables with $F^{-1}(d_1, d_2)$ and $F(d_1, d_2)$ factorable into [§28 (11)]

I. $(d_1 - \alpha d_2) f(d_1, d_2)$;

$$PF \equiv \frac{1}{d_2 f(\alpha, 1)} \frac{1}{d_1 - \alpha d_2}$$

II. $(d_1 - \alpha d_2)^p \cdot f(d_1, d_2)$;

$$\sum_{i=1}^{p} \frac{N_i(\alpha)}{d_2^{k-i}} \cdot \frac{1}{(d_1 - \alpha d_2)^i}$$

$k$ the degree of $f$ in $d_2$

III. $[(d_1 - \alpha d_2)^2 + \beta^2 d_2^2] \cdot f(d_1, d_2)$;

$$\frac{L d_1 + M d_2}{d_2^{k-2}[(d_1 - \alpha d_2)^2 + \beta^2 d_2^2]}$$

IV. $[(d_1 - \alpha d_2)^2 + \beta^2 d_2^2]^p \cdot f(d_1, d_2)$;

$$\sum_{i=1}^{p} \frac{L_i d_1 + M_i d_2}{d_2^{k-2}[(d_1 - \alpha d_2)^2 + \beta^2 d_2^2]^i}$$

Fundamental theorems: [§29 (1)]

I. $F(d_1, d_2) \cdot e^{\phi(x,y)}$

$$\equiv e^{\phi(x,y)} F\left[d_1 + \frac{\partial\phi}{\partial x}, d_2 + \frac{\partial\phi}{\partial y}\right]$$

II. $e^{\phi(d_1,d_2)} F(x, y)$

$$\equiv F\left[x + \frac{\partial\phi}{\partial d_1}, y + \frac{\partial\phi}{\partial d_2}\right] e^{\phi(d_1,d_2)}$$

III. $F(d_1, d_2)\phi(ax + by)$

$$\equiv F(a, b)\phi^{(n)}(ax + by)$$

$$\text{IV. } F(d_1, d_2) \equiv e^{p_1\frac{\partial}{\partial q_1}+p_2\frac{\partial}{\partial q_2}} \cdot F(q_1, q_2)$$
$$\equiv \sum_{k=0}^{\infty}\frac{1}{k!}\left[p_1\frac{\partial}{\partial q_1} + p_2\frac{\partial}{\partial q_2}\right]^k F(q_1, q_2)$$

Homogeneous inverses:

$$(d_1 - md_2)^{-1} \equiv e^{mxd_2}d_1^{-1}e^{-mxd_2} \equiv \mu_1^{-1}e^{m(x-u)d_2} \qquad [\text{VII } \S 30 \ (3)]$$
$$\equiv \int^x e^{m(x-u)d_2}()\partial u$$
$$(d_1 - md_2)^{-p} \equiv e^{mxd_2}d_1^{-p}e^{mxd_2} \equiv \mu_1^{-p}e^{m(x-u)d_2} \qquad (4)$$
$$\equiv \int^x\int^x \cdots \int^x e^{m(x-u)d_2}()\partial u^p$$
$$\equiv \mu_1^{-1}\frac{(x-u)^{p-1}}{(p-1)!}e^{m(x-u)d_2}$$
$$\equiv \int^x\frac{(x-u)^{p-1}}{(p-1)!}e^{m(x-u)d_2}()\partial u$$
$$(d_1^2 + \beta^2d_2^2)^{-1} \equiv \beta^{-1}d_2^{-1}\mu^{-1}\sin\beta(x-u)d_2 \qquad (5)$$
$$\equiv \beta^{-1}\int^y\partial y\int^x \sin\beta(x-u)()\partial u$$
$$[(d_1 - \alpha d_2)^2 + \beta^2d_2^2]^{-1}$$
$$\equiv e^{\alpha xd_2}(d_1^2 + \beta^2d_2^2)^{-1}e^{-\alpha xd_2}$$
$$[\text{see VII } \S 30 \ (5)]$$
$$(d_1^2 + \beta^2d_2^2)^{-p} \equiv \beta^{-p}[d_2^{-1}\mu^{-1}\sin\beta(x-u)d_2]^p \qquad (6)$$
$$[(d_1 - \alpha d_2)^2 + \beta^2d_2^2]^{-p}$$
$$\equiv e^{\alpha xd_2}(d_1^2 + \beta^2d_2^2)^{-p}e^{-\alpha xd_2}$$
$$[\text{see VII } \S 30 \ (6)]$$

Nonhomogeneous inverses:

$$(d_1 - md_2 - a)^{-1} \equiv e^{mxd_2+ax}d_1^{-1}e^{-mxd_2-ax} \qquad (9)$$
$$\equiv \mu^{-1}e^{(x-u)(md_2+a)}$$
$$\equiv \int^x e^{(x-u)(md_2+a)}()\partial u$$
$$(d_1 - md_2 - a)^{-p} \qquad (10)$$
$$\equiv e^{x(md_2+a)}d_1^{-p}e^{-x(md_2+a)}$$
$$\equiv \mu^{-p}e^{(x-u)(md_2+a)}$$
$$\equiv \int^x\int^u \cdots \int^u e^{(x-u)(md_2+a)}()\partial u^p$$
$$\equiv \mu^{-1}\frac{(x-u)^{p-1}}{(p-1)!}e^{(x-u)(md_2+a)}$$
$$\equiv \int^x\frac{(x-u)^{p-1}}{(p-1)!}e^{(x-u)(md_2+a)}()\partial u$$

$$[(d_1 - \alpha d_2)^2 + \beta^2]^{-1} \equiv e^{\alpha x d_2}(d_1{}^2 + \beta^2)^{-1}e^{-\alpha x d_2}$$ [VII §30 (11)]

$$\equiv (\mu_1{}^2 + \beta^2)^{-1}e^{\alpha(x-u)d_2}$$

$$[(d_1 - \alpha d_2)^2 + \beta^2]^{-1} \equiv (\mu_1{}^2 + \beta^2)^{-p}e^{\alpha(x-u)d_2} \quad (12)$$

$$(d_1 - \alpha)^2 + (d_2 - \beta)^{-1}$$

$$\equiv e^{\alpha x + \beta y}(d_1{}^2 + d_2{}^2)^{-1}e^{-(\alpha x + \beta y)}$$

$$\equiv (\mu_1{}^2 + \mu_2{}^2)^{-1}e^{\alpha(x-u_1)+\beta(y-u_2)} \quad (13)$$

Homogeneous operations on zero: [VII §31]

$$d_1{}^{-1} \cdot 0 = \phi(y)$$

$$d_2{}^{-1} \cdot 0 = f(x) \quad (2)$$

$$d_1{}^{-n} \cdot 0 = \sum_{k=0}^{n-1} x^k \phi_k(y)$$

$$d_2{}^{-n} \cdot 0 = \sum_{k=0}^{n-1} y^k f_k(x) \quad (3)$$

$$d_1{}^{-m}d_2{}^{-n} \cdot 0$$

$$= \sum_{k=0}^{m-1} x^k \phi_k(y) + \sum_{k=0}^{n-1} y^k f_k(x) \quad (4)$$

$$(d_1 - \alpha d_2)^{-1} \cdot 0 = \phi(y + \alpha x)$$

$$(d_1 - \alpha d_2)^{-p} \cdot 0 = \sum_{k=0}^{p-1} x^k \phi(y + \alpha x) \quad (5)$$

$$\prod_{k=1}^{n} (d_1 - \alpha_k d_2)^{-1} \cdot 0 = \sum_{k=1}^{n} \phi_k(y + \alpha_k x) \quad (6)$$

$$(d_1{}^2 + \beta^2 d_2{}^2)^{-1} \cdot 0$$

$$= \phi_1(y + i\beta x) + \phi_2(y - i\beta x)$$

$$= (\cos \beta x d_2)\phi_1(y) + (\sin \beta x d_2)\phi_2(y) \quad (7)$$

$$(d_1{}^2 + \beta^2 d_2{}^2)^{-p} \cdot 0$$

$$= \sum_{k=0}^{p-1} x^k[\phi_{1k}(y + i\beta x) + \phi_{2k}(y - i\beta x)]$$

$$= (\cos \beta x d_2)\left[\sum_{k=1}^{p} x^k A_{1k}(y)\right] + (\sin \beta x d_2)\left[\sum_{k=1}^{p} x^k A_{2k}(y)\right] \quad (8)$$

$$[(d_1 - \alpha d_2)^2 + \beta^2 d_2{}^2]^{-1} \cdot 0$$

$$= e^{\alpha x d_2}(d_1{}^2 + \beta^2 d_2{}^2)^{-1} \cdot 0$$

[see VII §31 (7)] (9)

$$[(d_1 - \alpha d_2)^2 + \beta^2 d_2{}^2]^{-p} \cdot 0$$

$$= e^{\alpha x d_2}(d_1{}^2 + \beta^2 d_2{}^2)^{-p} \cdot 0$$

[see VII §31 (8)]

$$\prod_{k=1}[(d_1 - \alpha_k d_2)^2 + \beta_k{}^2 d_2{}^2]^{-1} \cdot 0$$

$$= \sum_{k=1} e^{\alpha_k x d_2}(d_1{}^2 + \beta_k{}^2 d_2{}^2)^{-1} \cdot 0 \quad \text{[VII §31 (10)]}$$

[see VII §31 (7)]

$$\prod[(d_1 - \alpha_k d_2)^2 + \beta_k{}^2 d_2{}^2]^{-p_k} \cdot 0$$

$$= \sum_k \left[ \sum_{j=1}^{p_k} e^{\alpha_k x d_2}(d_1{}^2 + \beta_k{}^2 d_2{}^2)^{-j} \right] \cdot 0 \tag{11}$$

[see VII §31 (8)]

Nonhomogeneous operations on zero:

(a) $(d_1 - md_2 - a)^{-1} \cdot 0 \equiv e^{x(md_2+a)} d_1{}^{-1} \cdot 0$ (12)

[see VII §31 (2)]

(b) $(d_1 - md_2 - a)^{-p} \cdot 0 \equiv e^{x(md_2+a)} d_1{}^{-p} \cdot 0$

[see VII §31 (3)]

(c) $[(d_1 - \alpha d_2)^2 + \beta^2]^{-1} \cdot 0$

$$\equiv e^{\alpha x d_2}(d_1{}^2 + \beta^2)^{-1} \cdot 0$$

[see VII §31 (5)]

(d) $[(d_1 - \alpha d_2)^2 + \beta^2]^{-p} \cdot 0$

$$\equiv e^{\alpha x d_2}(d_1{}^2 + \beta^2)^{-p} \cdot 0$$

[see VII §31 (8)]

(e) $[(d_1 - \alpha)^2 + (d_2 - \beta)^2]^{-1} \cdot 0$

$$\equiv e^{\alpha x + \beta y}(d_1{}^2 + d_2{}^2)^{-1} \cdot 0$$

[see VII §31 (7)]

(f) $[(d_1 - \alpha)^2 + (d_2 - \beta)^2]^{-p} \cdot 0$

$$\equiv e^{\alpha x + \beta y}(d_1{}^2 + d_2{}^2)^{-p} \cdot 0$$

[see VII §31 (8)]

Homogeneous operations on unity: [§32]

$$d_1{}^{-p} \cdot 1 = \frac{x^p}{p!}, \qquad d_2{}^{-p} \cdot 1 = \frac{y^p}{p!} \tag{1}$$

$$(d_1 - \alpha d_2)^{-1} \cdot 1 = -\frac{1}{2\alpha}(y - \alpha x) \tag{3}$$

$$(d_1 - \alpha d_2)^{-p} \cdot 1 = \frac{1}{(-2\alpha)^p p!}(y - \alpha x)^p \tag{4}$$

$$\prod_{=1k}^{m} (d_1 - \alpha_k d_2)^{-1} \cdot 1 = \sum_{k=1}^{m} \frac{1}{-2\alpha_k}(y - \alpha_k x) \tag{5}$$

$$\prod_{k=1}^{m}(d_1 - \alpha_k d_2)^{-p_k} \cdot 1 = \frac{1}{z!}\left[x^z + \prod_{k=1}^{m}(-\alpha_k)^{p_k} y^z\right] \tag{6}$$

$$z = \prod_{k=1}^{m} p_k$$

$$(d_1^2 + \beta^2 d_2^2)^{-m} \cdot 1 = \frac{1}{2\beta^{2m}(2m)!}[y^{2m} + (\beta x)^{2m}] \qquad \text{[VII §32 (7)]}$$

$$[(d_1 - \alpha d_2)^2 + \beta^2 d_2^2]^{-m} \cdot 1$$

$$= \frac{1}{2(\alpha^2 + \beta^2)^m (2m)!}[y^{2m} + (\alpha^2 + \beta^2)^m x^{2m}]$$

$$\prod_{k=1}^{m}[(d_1 - \alpha_k d_2)^2 + \beta_k^2 d_2^2]^{-p_k} \cdot 1$$

$$\frac{1}{2\prod_{k=1}^{m}(\alpha_k^2 + \beta_k^2)^{p_k}(2\Sigma p_k)!}\left[y^{2\Sigma p_k} + x^{2\Sigma p_k}\prod_{k=1}^{m}(\alpha_k^2 + \beta_k^2)^{p_k}\right]$$

The single partial L.D.E. with C.C.: [VIII §36]

$$F(d_1, d_2) \cdot z = f(x, y) + 0$$

$$y = F^{-1}(d_1, d_2) \cdot f(x, y) + F^{-1}(d_1, d_2) \cdot 0 \tag{1}$$

Systems of P.D.E. with C.C. [§37]

$$\sum_{j=1}^{m} F_{ij}(d_1, d_2) \cdot z_j = X_i(x, y) \qquad i = 1, 2, \cdots, m$$

Theorem I: $| F_{ij}(d_1, d_2) | \cdot V = 0$

Characteristic equation $| F_{ij}(d_1, d_2) | = 0$ (4)

Theorem II: $z_j = f_{ij}(d_1, d_2) \cdot V$ (5)

$$s + p = p \cdot q \tag{8}$$

Cramer's rule: $z_j = | F_{ij}(d_1, d_2) |^{-1} \cdot K_j$ (10)

$$d_i^{-1} \equiv \int^{x_i}()\partial x_i + \phi(x_j), \qquad j = 1, 2, \cdots, n \ \ (\neq i) \qquad \text{[IX §38 (1)]}$$

$$\equiv \int_a^{x_i}()\partial x_i$$

Fundamental theorems in $n$ variables: [IX §38 (2)]

I. $F(d_i)e^{\phi(x_i)} \equiv e^{\phi(x_i)}F\left[d_i + \dfrac{\partial\phi}{\partial x_i}\right]$

$$\text{II. } e^{\phi(d_i)}f(x_i) \equiv f\left[x_i + \frac{\partial\phi}{\partial d_i}\right]e^{\phi(d_i)}$$

$$\text{III. } F(\pi)\cdot\Theta_m \equiv \Theta_m \cdot F(\pi + m)$$

$$\pi \equiv \sum_{i=1} a_i d_i$$

$$\text{IV. } F_k(d_i)\phi(\pi) = F_k(a_i)\phi^{(k)}(\pi)$$

$$\text{V. } F(d_i) \equiv \left[exp\left(\Sigma p_i \frac{\partial}{\partial q_i}\right)\right]\cdot F(q_i)$$

$$\equiv \sum_{n=0}^{\infty}\frac{1}{n!}\left[\sum_i p_i \frac{\partial}{\partial q_i}\right.\cdot F(q_i)$$

$$exp\left|\sum_{i=1}\left(\sum_{\substack{j=1\\ \neq s}} p_{ij}\right)\frac{\partial}{\partial p_{si}}\right. F(p_{si})$$

The noncommutative operator $\vartheta \equiv xD$ [X §40]

$$(A)\ \vartheta \equiv xD, \qquad x = e^z, \qquad D \equiv \frac{d}{dx}, \qquad \vartheta \equiv \frac{d}{dz} \tag{1}$$

$$\vartheta \equiv f'[\phi(x)]\cdot D, \qquad x = f(z), \qquad z = \phi(x)$$

$$(B)\ Dx \equiv xD + 1 \tag{3}$$

Fundamental theorems:

I–VI. For each set in $(x, D)$,

a similar set in $(z, \vartheta)$ (6)

$$\text{VII. } \vartheta^n \equiv \sum_{k=n}^{1} A_k x^k D^k$$

$$\text{VIII. } x^n D^n \equiv \prod_{\alpha=0}^{n-1}(\vartheta - \alpha) \equiv \sum_{k=n}^{1} B_k \vartheta^k$$

$$\text{IX. } F(\vartheta)x^m \equiv x^m F(\vartheta + m)$$

$$\text{X. } F(\vartheta)\cdot\phi(x) \equiv \sum_{k=0}^{\infty}\frac{1}{k!}[F^{(k)}(\vartheta)\cdot\phi(x)]\vartheta^k$$

Stirling numbers: [X §40]

$$a_{m+1,\,k+1} = (m - k + 1)a_{mk} + a_{m,k+1} \tag{7}$$

$$a_{m+1,\,k+1} = -ma_{mk} + a_{m,k+1}$$

$$e^{k\vartheta}\cdot f(x) = f(xe^k) \tag{13}$$

$$e^{kx\vartheta}\cdot f(x) = f\left(\frac{x}{1 - kx}\right)$$

Fundamental theorems for solutions in series of linear differential equations: [XI §45]

I. $\Phi \cdot F \equiv 1$

$$\Phi \equiv \sum_k \phi_k, \qquad F \equiv \sum_k F_k$$

$$\sum_{i=0}^{k} F_i \phi_{k-i} = 0 \tag{2}$$

II. $F_k = (-1)^k \dfrac{1}{\phi_0{}^k}$

$$\begin{vmatrix} \phi_1 & \phi_0 & \cdot & \cdot \\ \phi_2 & \phi_1 & \phi_0 & \cdot \\ \phi_3 & \phi_2 & \phi_1 & \phi_0 \\ & & & \\ \phi_k & \phi_{k-1} & & \end{vmatrix}$$

III. Max Mason's theorem:

$$f = g + S \cdot f,$$
$$(1 - S)f = g.$$

IV. Boole's theorem:

$$\sum_{k=0}^{n} f_k(\vartheta) e^{kz} \cdot u = 0$$

$$u = \sum_{k=0}^{\infty} F_k \cdot \left[ \frac{1}{f_i(\vartheta)} 0 \right] \qquad \text{for any } i$$

# APPENDIX III

## §59. Historical Note.[1]

The name "symbolic calculus" is given to those operations in the differential and integral calculus and in differential equations which are indicated by the use of letters whose combinations follow certain of the laws of algebra. Robert Carmichael and others call them the "calculus of operations"; Salvatore Pincherle speaks of them under the name "functional operations"; and George Boole uses them in his book on differential equations under the title "symbolic methods"; while Salvatore Pincherle also calls them the "distributive operation."

The historical development of these operations or methods is very interesting and seems worthy of being set down in English. There are two foreign sources and two British where the main outline of what follows may be secured, but they are either not readily available or out of print, and moreover they are not complete or entirely accurate.

All the methods of the symbolic calculus appear to have originated from the symbol by which G. A. Leibnitz represented the operation of differentiation, *viz.*, $d$, $d^2$, $d^3$, etc. He first recognized an analogy between the successive differential terms of Taylor's series and the terms of a binomial expansion. Under date of May 16, 1695, we find him writing to Jean Bernoulli as follows:

> Multa adhuc in istis summerum & differentiarum progressionibus latent, quae paulatim prodibunt. Ita notabilis est consensus inter numeros potestam a binomio & differentiarum rectanguli; & puto nescio quid arcani subesse.

To this under date of June 18 Bernoulli replied:

> Nihil elegantius est quam consensus quem observasti inter numeros potestam a binomio & differentiarum rectangulo; haud dubie aliquid arcani subest.[2]

[1] From an article by the author on "Symbolic Calculus" in the *Washington University Studies, sci. ser.*, **12,** No. 2 (1925), 137–141.

[2] (G. A. L. to J. B.) Many things thus far lurking in these summations and progressive differentials will gradually come forth. Thus notably is

They, as well as J. L. Lagrange, A. M. Lorgna, and J. Ph. Grüson, entered heartily into the discussion about this analogy, but it was reserved for L. F. A. Arbogast, who was then professor of mathematics at the University of Strasbourg, in the year 1800 to suggest the "method of the separation of the chain of operations" from the thing operated upon and to substitute a capital $D$ for the little $d$ of Leibnitz. He says:

J'applique aux différéntielles, aux dérivées générales, aux relations entre les différéntielles et les différences (finies), une méthode de calcul qu'on peut nommer **méthode de séparation des échelles d'opération**: elle donne le moyen de présenter sous une forme três simple des formules compliquées et de parvenir avec une extrême facilité à des résultats importans. Considérée généralement, cette méthode consiste à détacher de la fonction des variables, lorsque cela est possible, les signes d'opération qui affectent cette fonction, et à traiter l'expression formée de ces signes mêlés avec des quantités quelconques expression que j'ai nomée **échelle d'opérations,** à la traiter, dis-je, tout de même que si les signes d'opérations qui y entrent étoient des quantités; puis à multiplier le résultat par la fonction. 'Preface'.

And again:

En désignant par $D$ l'opération qu'l faut faire sur $F$ a pour en déduire $b$, il est clair qu'on aura $b = DFa$, $c = D \cdot DFa = D^2Fa$, $d = D^3Fa$, etc. Ainsi la série précédente $\left[F(a+x) = a + bx + \frac{c}{1.2}x^2 + \frac{d}{1.2.3}x^3 + \text{etc.}\right]$ peut aussi s'exprimer de cette manière:

$$F(a+x) = Fa + \frac{DFa}{1}x + \frac{D^2Fa}{1.2}x^2 + \quad \text{etc.} \qquad \text{[page 2]}$$

Of Arbogast's work we have appraisal by Samuel Roberts, in which he says:

The work of Arbogast . . . is so remarkable for the generality and power of its processes, and for the distinct expression of the fundamental principle of the calculus of operations, a method nonexistent at that

the agreement between the power numbers of the binomial and the differentials, and I believe I do not know what is hidden there.

(J. B. to G. A. L.) Nothing is more elegant than the agreement which you have observed between the power numbers of the binomial and the differentials; without doubt there is something hidden in it.

time in any systematized form, that it seems right still to connect his name with the general subject of development.

It appears to me that the distinguishing feature . . . is the substitution of an operation on one set of letters (or quantities symbolized by them) for an equivalent operation on another set of letters (or quantities symbolized by them). This is the doctrine of equivalent operators. . . .

Arbogast did not successfully and correctly actually separate the symbols of operation from the variable, but this was done by his son-in-law M. J. F. Français, in 1811, and also a year later by F. Servois, who Gregory incorrectly says was the first to do it. Witness the actual example by which M. Français illustrated this separation: For

$a \cdot F(x, y) + b \cdot F(x, y) + \cdots = (a + b + c + \cdots)F(x, y)$ [he put] $f_1(a, b, c \cdots)F(x, y) + f_2(a, b, c \cdots)F(x, y) + \cdots = (f_1 + f_2 + \cdots)F(x, y)$.

This practically closes the first period, the origin of the symbolic calculus, for it is worthy of note that so far as the symbolic character of the operations is concerned nothing extensive further appears in the French journals for many years; and although the theory of linear differential equations and that of finite differences have profited immensely by the later developments, the French seem to have ignored the symbolic method almost entirely. Of this attitude D. F. Gregory says:

Fourier, in his "Traité de Chaleur," published in 1822, has shown that the series which are obtained in the solution of several partial differential equations may be conveniently expressed by the separation of symbols of operation from those of quantity. . . . Other French writers seem to have avoided carefully entering at all on the track which Fourier opened. . . .

Mr. Greatheed, in a paper published in the number of the *Philosophical Magazine* for September, 1837, was, I believe, the first to call the attention of mathematicians to the utility of this method in the case of partial differential equations, but he had not then reduced it to its greatest degree of simplicity.

The **second period** was ushered in by Greatheed with the article mentioned above by Gregory. Three distinct lines of development have to be noted—that leading to the theory of

finite differences, that to fractional or general differentiation (or, rather, to the use of symbolic methods in fractional differentiation, for it was Liouville who originated that subject), and that to symbolic methods in the calculus and in differential equations. It is particularly with respect to the two latter that my study has been made, and I am neglecting entirely the development of finite differences. Papers appeared rapidly between 1836 and 1860, and a complete body of theory was built up. Two textbooks embodying it were in print by 1860—one by Robert Carmichael, and one by George Boole, with pretty nearly all the theorems in the shape that we now have them. In 1866, G. S. Carr published his "Synopsis of Elementary Results in Pure Mathematics," in which he listed under symbolic methods all the theorems by all the writers prior to that date on that subject but not disclosing to whom each was to be credited. Up to the end of the century more than one hundred papers had appeared. It is not my purpose in this section to analyze the contributions made by each writer. I wish, however, to note briefly those who made the largest contributions.

As will be seen from the bibliography, D. F. Gregory, Robert Carmichael, and George Boole wrote the largest number of, and the most voluminous, papers, and each is credited with a textbook. Other distinguished names are Arthur Cayley, J. J. Sylvester, Augustus De Morgan, J. W. L. Glaisher, Oliver Heaviside, and George Chrystal. The last two names deserve special mention because of the epoch-making character of their work.

In 1892–1898, there appeared in the *Electrician*, of London, Oliver Heaviside's applications of the calculus of operations to electromagnetic problems. These are peculiarly interesting for the boldness and freedom with which he uses the symbolic methods and fractional differentiation and the remarkable practical results that he obtained. It appears, however, that he was either not aware of the complete body of theory on this subject which had been in existence more than twenty years before he wrote or he drew freely, though inadequately, upon it and gave no one credit. That he drew inadequately seems to be true, for we find him complaining bitterly of the severe criticism of his lack of rigor by the mathematicians of his day.

In 1897, George Chrystal tried to remedy some of the defects of the theory, especially as regards the equivalence of systems of ordinary linear differential equations. His paper closes the second period in the development of the subject.

The **third period** I should call that of the practical application of the operational theory by a wide group of electrical engineers. The "Heaviside method" became the rage. Many hastened to "explain" his "shifting theorem" and his "expansion theorem." Quite a number of papers appeared during the years 1910 and 1925 on the "Heaviside operational calculus." A significant feature of these papers was the lack of a full bibliography of the subject. Prominent in this were John R. Carson, Louis Cohen, E. J. Berg, H. W. March, W. O. Pennell, and J. J. Smith. This period ended with the publication of three books, *viz.*, those of John R. Carson in 1926 and of E. J. Berg and Vannevar Bush each in 1929.

The last decade has seen the entry of the mathematicians into the field of the exegesis of operational methods. Harold Jeffries with his "Operational Methods in Mathematical Physics," Cambridge, 1927; Paul Levy, "Le Calcul Symbolique D'Heaviside," 1926, F. D. Murnaghan in 1927 and 1928; Norbert Wiener, 1926, Burchnall and Chaundy, H. Bateman, J. V. Neumann, H. S. Carslaw, E. L. Post, H. T. Davis, and others have endeavored to call attention to the necessity of rigor in the use of the operational or "symbolic calculus," with more or less success.

The difficulty has seemed to the author to be the general lack of knowledge of what the literature has to offer and the correlation and extension of that large body of information and basic work.

This text has aimed to bring that basic work into the open, elucidate it in as simple a manner as possible, fill in the gaps, and extend it to as wide a field of application as possible. A large part of this text is new. The examples have been drawn from many fields. It, however, is but a beginning, and it is the author's hope that the whole subject will eventually become of considerable use to all workers in the field of applied mathematics. Below is given a complete bibliographical list of all papers and works on operational methods and applications from their beginnings in 1765 to date.

Since this text is based on the work already in the literature from 1800 to 1900, it is only just that the fundamental theorems of the originators of the methods should be exhibited here. This is the more necessary because their work shows the operational method in its virgin purity, entirely uncontaminated by modern fads; and to understand the method properly its study should be free from the nonoperational or pseudo-operational methods employed in prior texts and in higher mechanics. The following three sections are abstracts of the work of three distinguished men, Robert Murphy, George Boole, and Charles Graves, whose work still can be drawn upon for further developments in operational method beyond even the limits of this text. The careful reading of these sections by the student is recommended.

**§60. Theorems of Robert Murphy.**

This section follows very closely some of the work done by Robert Murphy in his First Memoir on the Theory of Analytical Operations, *Phil. Trans. London* (1837), pp. 179–210. The notation has been modernized, and the arrangement has been changed to become consistent with the rest of this text. Additions and simplifications have been made wherever possible. The section, however, is Murphy's and no modern analysis has in any way altered the fundamental validity of his work.

(1) Linear operators are distributive:

$$\begin{aligned} p(a+b) &= p\cdot a + p\cdot b \\ \psi[f(x)+\phi(x)] &= f(x+h)+\phi(x+h) \\ &= \psi\cdot f(x)+\psi\cdot\phi(x) \end{aligned}$$

(2) Polynomials, compounds, or any function of linear operators are linear and thus distributive:

$$\begin{aligned} (D+d)(X+R) &= (D+d)X+(D+d)R \\ D\cdot\Delta(X+R) &= D\cdot\Delta\cdot X + D\cdot\Delta\cdot R \end{aligned}$$

(3) The composition of polynomial operators is affected as in algebraic multiplication; but when noncommutative, order must be preserved.

If $A$, $B$, $H$, and $K$ are linear operators, then

$$\begin{aligned} (A+B)(H+K) &\equiv (A+B)H+(A+B)K \\ &\equiv AH+BH+AK+BK \end{aligned}$$

$$(A + B)^2 \equiv (A + B)(A + B)$$
$$\equiv (A + B)A + (A + B)B$$
$$\equiv A^2 + BA + AB + B^2, \qquad \text{if} \qquad AB \not\equiv BA$$

but

$$A^2 + 2AB + B^2, \qquad \text{if} \qquad AB = BA$$

$$(A + B)^3$$
$$\equiv (A + B)(A + B)(A + B)$$
$$\equiv (A^2 + AB + BA + B^2)(A + B)$$
$$\equiv A^3 + ABA + BA^2 + B^2A + A^2B + AB^2 + BAB + B^3$$

Placing

$$BA^{(2)} \equiv A^2B + ABA + BA^2$$

and

$$B^{(2)}A \equiv AB^2 + BAB + B^2A$$

we obtain

$$(A + B)^3 \equiv A^3 + BA^{(2)} + B^{(2)}A + B^3, \qquad \text{when} \qquad AB \not\equiv BA$$

but

$$\equiv A^3 + 3A^2B + 3AB^2 + B^3, \qquad \text{when} \qquad AB \equiv BA$$

And in general, similarly,

$$(A + B)^n \equiv \sum_{k=n}^{0} B^{(n-k)} \cdot A^{(n)}, \qquad \text{when} \qquad AB \not\equiv BA$$

$$\equiv \sum_{k=n}^{0} {}_nC_k \cdot B^{n-k} \cdot A^n, \qquad \text{when} \qquad AB \equiv BA$$

where ${}_nC_k \equiv \dfrac{n!}{(n-k)!k!}$.

(4) From (3) we can immediately obtain expansions of two well-known operators $\Delta$ and $\psi$.

*a.* Defining $\Delta$ by $\Delta \cdot f(x) = f(x + h) - f(x)$, and setting $\psi \equiv \Delta + 1$, we shall have

$$\psi \cdot f(x) = (\Delta + 1)f(x) = \Delta \cdot f(x) + f(x)$$
$$= f(x + h) - f(x) + f(x) = f(x + h)$$

which defines $\psi$.

*b.* Then

$$\Delta \equiv \psi - 1$$

and

$$\Delta^n \equiv (\psi - 1)^n$$

$$\equiv \sum_{k=n}^{0} (-1)^{n-k} \cdot {}_nC_k \cdot \psi^k, \qquad \text{since} \qquad \psi \cdot 1 \equiv 1 \cdot \psi$$

*c.*

$$\psi \equiv \Delta + 1$$

$$\psi^n \equiv (\Delta + 1)^n$$

$$\equiv \sum_{k=0} {}_nC_k \cdot \Delta^k, \qquad \text{since} \qquad \Delta \cdot 1 \equiv 1 \cdot \Delta$$

*d.* Operating on $f(x)$ by $\psi^n$,

$$\psi^n \cdot f(x) = \sum_{k=0}^{n} {}_nC_k \cdot \Delta^k \cdot f(x) = f(x + nh)$$

[Taylor's th. for finite differences]

*e.* In (*d*) put $nh = k$; we have

$$f(x + k) = \sum_{i=0}^{n} {}_{k/h}C_i \cdot \Delta^i \cdot f(x)$$

$$= \sum_{i=0}^{n} {}_kC_{ih} \cdot \left(\frac{\Delta}{h}\right)^i \cdot f(x)$$

But

$$\underset{h \to 0}{\mathrm{L}} \frac{\Delta}{h} u_x = \underset{h \to 0}{\mathrm{L}} \frac{u_{x+h} - u_x}{h} = \frac{d}{dx} \cdot u = D \cdot u,$$

which applied here, together with letting $n \to \infty$ with $nh = k$ a constant, gives

$$f(x + k) = \sum_{i=0}^{\infty} \frac{k^i}{i!} D^i \cdot f(x)$$

[Taylor's th. for the differential calculus]

*f.* We can easily derive what other forms the operators $\psi$ and $\Delta$ must have from the foregoing theorem. Let $h = k$, and omit the subject:

$$\psi \equiv \sum_{i=0} \frac{h^i}{i!} D^i \equiv e^{hD}$$

and

$$\Delta \equiv \psi - 1 \equiv e^{hD} \quad 1 \equiv \sum_{i=1}^{\infty} \frac{h^i}{i!} D^i$$

(5) Immediately are deducible some of the properties of $\psi$:

a. $e^{hD} \cdot e^{kD} \equiv e^{(h+k)D}$, since $h$, $k$, and $D$ are commutative

b. $\prod_k e^{H_k} \equiv e^{\sum_k H_k}$ $H_k$ linear operators

c. Applying these to the subjects $f(x, y)$ and

$$f(x_1, x_2, x_3 \cdots) \equiv f(x_i),$$

respectively, we shall have

$$e^{hd_1+kd_2} \cdot f(x, y) = f(x + h, y + k) \quad \text{[Taylor's th. for two variables]}$$
$$e^{\sum_i h_i d_i} \cdot f(x_i) = f(x_i + h_i) \quad \text{[Taylor's th. for many variables]}$$

(6) *Inverse* operations lead to the introduction of an *appendage*, which when operations are linear must be annexed to the result to give it the most general value to which it is susceptible, for the inverses of such operators are themselves linear.

With $H(X + P) = H \cdot X + H \cdot P$

set

$$H \cdot X = X_1, \qquad H \cdot P = P_1$$

Then

$$H^{-1}(X_1 + P_1) = H^{-1} \cdot X_1 + H^{-1} \cdot P_1$$

showing the linearity of $H^{-1}$. Now, suppose the nature of $H$ to be such that $H \cdot P = 0$. Then if

$$H \cdot X = y$$
$$H(X + P) = y$$

Hence

$$H^{-1} \cdot y = X + P$$

Rather

$$H^{-1}(y + 0) = H^{-1}y + H^{-1} \cdot 0$$

The appendage therefore in a linear operation is the result of its action on zero. $P$ will express a *form*, but its magnitude must be susceptible of an infinity of values; *i.e.*, it contains arbitrary

constants which enter as multipliers, for if $A$ is such a constant, then, in general,

$$H \cdot A \cdot P \equiv A \cdot H \cdot P$$

and if $P = 0$, we have

$$H \cdot [A \cdot 0] \equiv A \cdot [H \cdot 0]$$

Therefore, whatever particular value may be assigned to $H \cdot 0$, a more comprehensive value is attained by its arbitrary multiplication by $A$.

$A$ is a multiplier when $X \equiv X(x)$, but it admits of more extended forms in case of several variables. Thus:

$$\psi(x) \cdot f(y) = f(y), \quad \text{or} \quad e^{hd_1} \cdot f(y) = f(y)$$
$$\Delta(x) \cdot f(y) = 0, \quad \text{or} \quad (e^{hd_1} - 1) \cdot f(y) = f(y) - f(y) = 0$$

Then if $X$ is any function of $x$, and $P$ any particular value of $\Delta^{-1}(x) \cdot X$, we shall have, more generally,

$$\Delta^{-1}(x) \cdot X = P + f(y)$$

or

$$\Delta^{-1}(x) \cdot [\phi(x) + 0] = P + f(y)$$

which includes the former.

(7) In compound operations, the appendage obtained by the first simple operation becomes a new subject for the succeeding operations, each of which may in like manner introduce a new appendage. Thus:

$$\begin{aligned} D^{-3} \cdot 0 &= D^{-2}[D^{-1} \cdot 0] \\ &= D^{-2}[C_2] = D^{-1}[D^{-1}C_2] \\ &= D^{-1}[C_2x + C_1] \\ &= D^{-1}C_2x + D^{-1}C_1 \\ &= C_2\frac{x^2}{2} + C_1x + C_0 \end{aligned}$$

(8) We give the derivation of two types of appendage and simply state others.

*a*. If $D^{-1} \cdot 0 = \phi(x)$

then

$$D \cdot \phi(x) = 0$$

and

$$D^n \cdot \phi(x) = 0$$

so that

$$\phi(x+h) = \phi(x) + hD \cdot \phi(x) + \frac{h^2}{2!}D^2 \cdot \phi(x) + \cdots$$
$$= \phi(x)$$

Setting $h = -x$, we have

$$\phi(0) = \phi(x)$$

or $\phi(x)$ is constant relative to $x$. Therefore, if $C_k$ be any arbitrary quantity independent of $x$ we have

$$D^{-1} \cdot 0 = C_k$$
$$D^{-2} \cdot 0 = C_k x + C_{k-1}$$

$$D^{-n} \cdot 0 \qquad \sum_{k=n-1} C_k \cdot x^k$$

*b*. It is easy to see that $\psi(x) \equiv e^{hd_1}$ taken directly or independently is incapable of introducing any appendage, for if

$$\psi^{-1}(x) \cdot 0 = \phi(x)$$

then

$$\psi(x) \cdot \phi(x) = 0 = \phi(x+h)$$

*i.e.*,

$$\phi(x) = 0$$

Therefore $$\psi^{-n}(x) \cdot 0 = 0$$

(*c*) $$\vartheta^{-n} \cdot 0 = \sum_{k=0}^{n-1} C_k (\log x)^k$$

(*d*) $$d_1^{-1} \cdot 0 = \phi(y)$$

(*e*) $$d_1^{-n} \cdot 0 = \sum_{k=0}^{n-1} x^k \cdot \phi_k(y)$$

(9) When the simple operators of a compound one are relatively commutative, their order of performance is indifferent; but when noncommutative, a transmutation of place (or change in order of performance) requires an alteration in the operators themselves. This is clearly seen from the following theorems.

(10) If $I$, $K$, and $H$ are operators connected by the relation

$$I \cdot K \equiv H \cdot I \tag{I}$$

then

$$I \cdot K^n \equiv H^n \cdot I \tag{II}$$

$$I \cdot K^{-n} \equiv H^{-n} \cdot I \tag{III}$$

$$I \cdot f(K) \equiv f(H) \cdot I \tag{IV}$$

For, by operating on the right of (I) by $I^{-1}$, we have

(*a*)
$$I \cdot K \cdot I^{-1} \equiv H \cdot I \cdot I^{-1} \equiv H$$

This is simply equivalent to supplying a subject $I^{-1} \cdot S$ for the identical operators of (I), using the identity $I \cdot I^{-1} \equiv 1$, and then abstracting the operators. Now, on (*a*) operate on the left by $H$, and substitute $H \cdot I$ for $I \cdot K$, obtaining

(*b*)
$$I \cdot K^2 \equiv H^2 \cdot I$$

Similarly, in general,

(*c*)
$$I \cdot K^n \equiv H^n \cdot I$$

Upon (*c*) operate on the left by $H^{-n}$, and on the right by $K^{-n}$, with $H^{-n}H^n \equiv K^{-n}K^n \equiv 1$; we obtain

(*d*)
$$I \cdot K^{-n} \equiv H^{-n} \cdot I$$

With (*c*) and (*d*), by taking various values of $n$, multiplying by the constants $a_n$, and adding, we easily find that

(*e*)
$$I \cdot f(K) \equiv f(H) \cdot I$$

(11) Thus if $I$ and $K$ are known, we can obtain $H$. A very interesting theorem can at once be derived.

*a*. With $K \equiv D$, $I \equiv e^{P(x)}$ in $I \cdot K \cdot I^{-1} \equiv H$, and subject $S$, we have

$$e^{P(x)} \cdot D \cdot e^{-P(x)} \cdot S = H \cdot S$$

But

$$D \cdot e^{-P(x)} \cdot S = e^{-P(x)} \cdot D \cdot S + S(-P'(x))e^{-P(x)} = e^{-P(x)}[D - P'(x)] \cdot S$$

so that

$$D - P'(x) \equiv H$$

and

$$e^{P(x)} \cdot f(D) \equiv f[D - P'(x)] \cdot e^{P(x)} \tag{V}$$

*b.* Setting $P(x) \equiv ax$ in (V), we have

$$e^{ax} \cdot f(D) \equiv f(D - a)e^{ax}*$$

(12) Again, with a subject $S \equiv f(x, y)$, $K \equiv D_1$, and $I \equiv e^{\phi(x)\frac{d}{dy}}$ defined by the equation

$$I \cdot f(x, y) \equiv f[x, y + \phi'(x)]$$

we should have

$$H \equiv I \cdot K \cdot I^{-1}$$

$$H \cdot S = e^{\phi(x)\frac{d}{dy}} \cdot D_1 \cdot e^{-\phi(x)\frac{d}{dy}} \cdot S$$

But

$$D \cdot e^{-\phi(x)\frac{d}{dy}} \cdot S = e^{-\phi(x)\frac{d}{dy}} \cdot D_1 \cdot S + \left[-\phi'(x)\frac{d}{dy}\right] e^{-\phi(x)\frac{d}{dy}} \cdot S$$

$$= e^{-\phi(x)\frac{d}{dy}} \cdot \left[D_1 - \phi'(x)\frac{d}{dy}\right] \cdot S$$

whence

$$H \equiv D - \phi'(x)\frac{d}{dy}$$

and

$$e^{\phi(x)d_2} f(d_1) \equiv f[d_1 - \phi'(x)d_2] \cdot e^{\phi(x)d_2} \qquad \text{(VI)}$$

where

$$d_1 \equiv \frac{\partial}{\partial x}, \qquad d_2 \equiv \frac{\partial}{\partial y}$$

or

$$I \cdot f(d_1) \equiv f[d_1 - \phi'(x)d_2] \cdot I$$

(13) The same method for many variables will give the form

$$I \cdot f(d_1) \equiv f\left[d_1 - \sum_i \phi'(x) \cdot d_i\right] \cdot I \qquad \text{(VII)}$$

where $I \equiv e^{\Sigma \phi_i(x) d_i}$,

and $d_1 \equiv \dfrac{\partial}{\partial x}$, $\quad d_i \equiv \dfrac{\partial}{\partial y_i}$

* This is the so-called "Heaviside shifting theorem," derived fifty years prior to its appearance in Heaviside's papers on electromagnetic theory.

(14) Examples of the use of the foregoing:

(a) $$D - a \equiv e^{ax} \cdot D \cdot e^{-ax}$$

(b) $$(D - a)^{-1} \equiv e^{ax} \cdot D^{-1} \cdot e^{-ax}$$

(c) $$\prod_{\alpha=0}^{n-1} (D - \alpha h) \equiv \prod_{\alpha=0}^{n-1} e^{\alpha h x} \cdot D \cdot e^{-\alpha h x}$$

(d) $$(a + b)^D f(x) \equiv f[x + \log (a + b)]$$

(e) $$\prod_{\alpha=0}^{n-1} (D - \alpha) e^{nx} \equiv (De^x)^n$$

(15) Thus when $I$ and $K$ are given, $H$ may generally be found, but it seemed to Murphy more difficult to discover $I$ when $K$ and $H$ are known. The latter problem was more easily solved by analysis developed by George Boole and Charles Graves, each from a different fundamental assumption, and given in the following sections.

## §61. Theorems of George Boole.

For his paper entitled On a General Method in Analysis, *Phil. Trans. London* (1844), pp. 225–283, George Boole was awarded one of the Royal Society's gold medals in that year. That paper was influenced by the criticism of David F. Gregory, the editor of the *Cambridge Mathematical Journal*, who most generously sent it to the Royal Society as more worthy of being published by them than by him. It illustrates the genius of Boole for digging down to fundamentals, which later showed up so brilliantly in his work on symbolic logic.

(1) Boole develops the elementary theory of operations from a single operation defined as follows: With $\rho$ and $\pi$ distributive operators and $\lambda$ a functional symbol operating on $\pi$ in such manner as $\lambda \cdot f(\pi) = f[\phi(\pi)]$, Boole writes

$$\rho \cdot f(\pi) \equiv \lambda \cdot f(\pi) \cdot \rho \tag{1}$$

Then

$$\rho^2 \cdot f(\pi) \equiv \lambda^2 \cdot f(\pi) \cdot \rho^2$$

$$\rho^n \cdot f(\pi) \equiv \lambda^n \cdot f(\pi) \rho^n \tag{2}$$

(2) Let us expand $f(\pi + \rho)$ in ascending powers of $\rho$.
Set

$$\pi + \rho \equiv \eta$$

Then

$$f(\pi + \rho) \equiv f(\eta)$$

Since $\eta$ is linear,

$$\eta \cdot f(\eta) \equiv f(\eta) \cdot \eta$$

or

$$(\pi + \rho) \cdot f(\pi + \rho) \equiv f(\pi + \rho) \cdot (\pi + \rho)$$

Now, assuming

$$f(\pi + \rho) \equiv \sum_m f_m(\pi) \cdot \rho^m$$

Then

$$\begin{aligned}(\pi + \rho) \cdot f(\pi + \rho) &\equiv (\pi + \rho) \cdot \sum_m f_m(\pi) \cdot \rho^m \\ &\equiv \pi \cdot \sum_m f_m(\pi) \cdot \rho^m + \rho \cdot \sum_m f_m(\pi) \cdot \rho^m \\ &\equiv \sum_m \pi \cdot f_m(\pi) \cdot \rho^m + \sum_m \rho \cdot f_m(\pi) \cdot \rho^m \\ &\equiv \sum_m \pi \cdot f_m(\pi) \cdot \rho^m + \sum_m \lambda \cdot f_m(\pi) \cdot \rho^{m+1} \quad (3)\end{aligned}$$

Also,

$$\begin{aligned}f(\pi + \rho) \cdot (\pi + \rho) &\equiv \sum_m f_m(\pi)\rho^m \cdot (\pi + \rho) \\ &\equiv \sum_m f_m(\pi)\rho^m \cdot \pi + \sum_m f_m(\pi)\rho^m \cdot \rho \\ &\equiv \sum_m f_m(\pi)\lambda^m\pi\rho^m + \sum_m f_m(\pi)\rho^{m+1} \quad (4)\end{aligned}$$

Equating coefficients of $\rho^m$ in (3) and (4) we have

$$f_m(\pi) \cdot \lambda^m\pi + f_{m-1}(\pi) = \pi f_m(\pi) + \lambda \cdot f_{m-1}(\pi)$$

from which

$$f_m(\pi) = \frac{(\lambda - 1) \cdot f_{m-1}(\pi)}{(\lambda^m - 1) \cdot \pi}$$

whence

$$f(\pi + \rho) \equiv \sum \frac{(\lambda - 1) \cdot f_{m-1}(\pi)}{(\lambda^m - 1) \cdot \pi} \cdot \rho^n$$

The first term is $f_0(\pi)$, or rather $K \cdot f_0(\pi)$. By using

$$f(\pi + \rho) \equiv \pi + \rho,$$

we can see that $K = 1$, and $f_0(\pi) = f(\pi)$

Now, if

$$\rho \cdot f(\pi) \equiv f(\pi + \Delta\pi) \cdot \rho$$

*i.e.*,

$$\lambda \cdot f(\pi) \equiv f(\pi + \Delta\pi)$$

we shall have

$$\begin{aligned}(\lambda - 1)f_{m-1}(\pi) &\equiv \lambda f_{m-1}(\pi) - f_{m-1}(\pi)\\ &\equiv f_{m-1}(\pi + \Delta\pi) - f_{m-1}(\pi)\\ &\equiv \Delta f_{m-1}(\pi)\\ (\lambda^m - 1)\pi &\equiv \lambda^m \cdot \pi - \pi\\ &\equiv \pi + m\Delta\pi - \pi\\ &\equiv m\Delta\pi\end{aligned}$$

And

$$f(\pi + \rho) \equiv \sum_m \frac{1}{m!} \cdot \left(\frac{\Delta}{\Delta\pi}\right)^m f(\pi) \cdot \rho^n \qquad \text{(I)}$$

(3) If in (I) $\Delta\pi \to 0$, then $\underset{\Delta\pi\to 0}{\mathrm{L}} \dfrac{\Delta}{\Delta\pi} = \dfrac{d}{d\pi}$, $\rho$ and $\pi$ become commutative, and (I) becomes

$$f(\pi + \rho) \equiv \sum_m \frac{1}{m!}\left(\frac{d}{d\pi}\right)^m f(\pi) \cdot \rho^m \qquad \text{[Taylor's th.]}$$

(4) If $\pi \equiv x$ and $\rho \equiv e^{rD}$ $\left(\text{where } D \equiv \dfrac{d}{dx}\right)$, then $x$ and $e^{rD}$ combine according to

$$e^{rD} f(x) \equiv f(x + r) \cdot e^{rD}$$

for, with a subject $S(x)$, we have

$$\begin{aligned}e^{rD} f(x) \cdot S(x) &= f(x + r) \cdot S(x + r)\\ &= f(x + r) e^{rD} S(x)\end{aligned}$$

Abstracting the subject, we have the theorem.

(5) If $\pi \equiv \dfrac{n\rho - x}{n}$ and $\rho = e^{rD}$

then

$$\frac{n\rho - x}{r} \text{ and } e^{rD} \text{ combine according to the theorem}$$

$$\rho \cdot f(\pi) \equiv f(\pi - 1) \cdot \rho$$

for

$$e^{rD} f\left(\frac{n\rho - x}{r}\right) \cdot S = f\left(\frac{n\rho - (x + r)}{r}\right) e^{rD} \cdot S$$
$$= f\left(\frac{n\rho - x}{r} - 1\right) e^{rD} S$$
$$= f(\pi - 1) e^{rD} S$$

whence the theorem by abstraction.

(6) If $\pi \equiv \frac{\rho - x}{r}$ and $\rho \equiv x e^{rD}$

Then

$$\rho \cdot f(\pi) \equiv f(\pi - 1) \cdot \rho$$

for

$$\rho \cdot f(\pi) \equiv x \cdot e^{rD} \cdot f\left(\frac{\rho - x}{r}\right)$$
$$\equiv x \cdot f\left[\frac{\rho - (x + r)}{r}\right] e^{rD}$$
$$\equiv f\left[\frac{\rho - x}{r} - 1\right] x \cdot e^{rD}$$
$$\equiv f(\pi - 1) \cdot \rho$$

By induction,

$$\rho^m \cdot f(\pi) \equiv f(\pi - m) \cdot \rho^m$$

Also,

$$\rho \cdot f(\pi + 1) \equiv f(\pi) \cdot \rho$$

. . . . . . . . . . . . .

$$\rho^m \cdot f(\pi + m) \equiv f(\pi) \cdot \rho^m \qquad \text{(II)}$$

(7) Expand $f(\pi + m)$ by Taylor's theorem:

$$f(\pi + m) \equiv f(m) + f'(m) \cdot \pi + \frac{f''(m)}{2!} \cdot \pi^2 +$$

with a subject $S \equiv 1$,

$$\pi \cdot 1 = \frac{x e^{rD} - x}{r} 1 = \frac{x - x}{r} = 0$$

$$\pi^n \cdot 1 = 0$$

so that

$$\rho^m \cdot f(\pi + m) \cdot 1 = \rho^m \cdot f(m)$$

and

$$\begin{aligned} f(\pi) \cdot \rho^m \cdot 1 &= \rho^m \cdot f(\pi + m) \cdot 1 \\ &= \rho^m \cdot f(m) \cdot 1 \\ &= f(m) \cdot \rho^m \cdot 1 \end{aligned}$$

By abstraction,

$$f(\pi) \cdot \rho^m \equiv f(m) \cdot \rho^m \tag{III}$$

(8) If $\pi \equiv \dfrac{\rho - x}{r}$ and $\rho \equiv x \cdot e^{rD}$,

then

$$\rho^{-1} \cdot \pi \equiv \rho^{-1} \cdot \frac{\rho - x}{\;} \equiv \frac{1 - \rho^{-1}x}{\;}$$

But

$$\begin{aligned} \rho^{-1} \cdot x &\equiv (x \cdot e^{rD})^{-1} \cdot x \\ &\equiv e^{-rD} \cdot x^{-1} \cdot x \equiv e^{-rD} \end{aligned}$$

Thus

$$\rho^{-1} \cdot \pi \equiv \frac{1 - e^{-}}{\;}$$

$$\cdots\cdots\cdots\cdots$$

$$(\rho^{-1} \cdot \pi)^n \equiv \left(\frac{1 - e^{-rD}}{r}\right)^n$$

Again,

$$\begin{aligned} (\rho^{-1} \cdot \pi)^2 &\equiv (\rho^{-1} \cdot \pi)(\rho^{-1} \cdot \pi) \\ &\equiv \rho^{-2}(\pi - 1)\pi \equiv \rho^{-2} \cdot \pi(\pi - 1) \\ &\cdots\cdots\cdots\cdots \\ (\rho^{-1} \cdot \pi)^n &\equiv \rho^{-n} \cdot \pi(\pi - 1)(\pi - 2) \cdots [\pi - (n - 1)] \end{aligned}$$

so that we shall have

$$\begin{aligned} (\rho^{-1} \cdot \pi)^n &\equiv \left(\frac{1 - e^{-rD}}{r}\right)^n \\ &\equiv \rho^{-n} \cdot \pi \cdot (\pi - 1)(\pi - 2) \cdots [\pi - (n - 1)] \end{aligned}$$

$$\therefore \quad \rho^n \left(\frac{1 - e^{-rD}}{r}\right)^n \equiv \pi(\pi - 1)(\pi - 2) \cdots [\pi - (n - 1)]$$

But

$$\begin{aligned} \rho &\equiv xe^{r\cdot} \\ \rho^2 &\equiv x \cdot e^{rD} \cdot x \cdot e^{rD} \equiv x(x + r)e^{2rD} \\ \rho^n &\equiv x \cdot (x + r)(x + 2r) \cdots [x + (n + 1)r]e^{nrD} \end{aligned}$$

We then have

$$x(x+r)(x+2r)\cdots[x+(n-1)r]e^{nrD}\left(\frac{1-e^{-rD}}{r}\right)^n \equiv \pi(\pi-1)(\pi-2)\cdots[\pi-(n-1)]$$

And as

$$e^{nrD}\left(\frac{1-e^{-rD}}{r}\right)^n \equiv \left[e^{rD}\cdot\frac{1-e^{-rD}}{r}\right]^n = \left(\frac{e^{rD}-1}{r}\right)^n$$

we shall have

$$\prod_{\alpha=0}^{n-1}(\pi-\alpha) \equiv \prod_{\alpha=0}^{n-1}(x+\alpha r)\left(\frac{e^{rD}-1}{r}\right)^n$$

which with $\dfrac{e^{rD}-1}{r} \equiv \dfrac{\Delta}{\Delta x}$ gives the theorem

$$\prod_{\alpha=0}^{n-1}(\pi-\alpha) \equiv \prod_{\alpha=0}^{n-1}(x+\alpha r)\left(\frac{\Delta}{\Delta x}\right)^n \tag{IV}$$

(9) Since $\operatorname*{L}_{\Delta x\to 0}\dfrac{\Delta}{\Delta x} = D$, and with $x \equiv e^z \equiv \rho$ and $\pi \equiv \dfrac{a}{dz} \equiv$

we shall have the following theorems:

$$f(\vartheta)\cdot e^{mz} \equiv e^{mz}f(\vartheta+m)^* \tag{V}$$

$$f(\vartheta)\cdot e^{mz}\cdot 1 = f(m)\cdot e^{mz} \tag{VI}$$

$$\prod_{\alpha=0}^{n-1}(\vartheta-\alpha) \equiv x^nD^n \tag{VII}$$

The rest of the earlier work of Boole deals with the fundamental theorems of operators which aid in the solution of differential equations.

**§62. Theorems of Charles Graves.**

Charles Graves was the Erasmus Smith Professor of Mathematics at Dublin University from 1843 to 1862 and Bishop of Limerick from 1866 to his death. The simplicity and power of

* See footnote to §61(11)*b*.

his mind are well attested by the following beautiful developments, covered in his paper printed in the *Proceedings of the Royal Irish Academy* for 1853–1857, pp. 144–152. It is well to quote the first two paragraphs from his paper.

(1) Let $\pi$ and $\rho$ be two distributive symbols of operation, which combine according to the law expressed by the equation

$$\rho \cdot \pi \equiv \pi \cdot \rho + a \tag{1}$$

$a$ being a constant or, at least, a symbol of distributive operation commutative with both $\pi$ and $\rho$.

(2) In this fundamental equation, if we change $\pi$ into $\rho$ and $\rho$ into $-\pi$ it [is unchanged]. From this it follows that in any symbolic equation

$$\phi(\pi, \rho) = 0 \tag{2}$$

which has been directly deduced from the fundamental equation (1), without any further assumption as to the nature of the operations denoted by $\pi$ and $\rho$, we may [deduce the correlative equation]

$$\phi(\rho, -\pi) = 0 \tag{3}$$

for this latter will be deducible from the primitive by the same processes.

(3) With $\rho \cdot \pi \equiv \pi \cdot \rho + a$,
then

$$\begin{aligned}
\rho \cdot \pi^2 &\equiv \pi \cdot \rho \cdot \pi + a\pi \\
&\equiv \pi(\pi \cdot \rho + a) + a\pi \\
&\equiv \pi^2 \cdot \rho + 2a\pi \\
\rho \cdot \pi^3 &\equiv \pi^3 \cdot \rho + 3a\pi^2
\end{aligned}$$

and

$$\rho \cdot \pi^n \equiv \pi^n \cdot \rho + na\pi^{n-1} \tag{4}$$

[for $n$ any positive integer]

(4) Again, $\rho \cdot \pi \equiv \pi \cdot \rho + a$.

$$\begin{aligned}
\pi^{-1} \cdot \rho \cdot \pi \cdot \pi^{-1} &\equiv \pi^{-1}(\pi\rho + a)\pi^{-1} \\
\pi^{-1} \cdot \rho &\equiv \rho\pi^{-1} + a\pi^{-2}
\end{aligned}$$

or

$$\rho\pi^{-1} \equiv \pi^{-1}\rho - a\pi^{-2}$$

Then

$$\rho\pi^{-2} \equiv \pi^{-2} \cdot \rho - 2a\pi^{-3},$$

and

$$\rho \cdot \pi^{-n} \equiv \pi^{-n}\rho - an\pi^{-n-1}$$

Thus Eq. (4) holds good for any integral value of $n$, positive or negative.

(5) Using Eq. (4) with various values of $n$ and multiplying by appropriate constants, we have

$$\begin{array}{ll} a_0 & \rho \equiv \rho \\ a_1 & \rho\pi \equiv \pi\rho + a \\ a_2 & \rho\pi^2 \equiv \pi^2\rho + 2a\pi \\ a_3 & \rho\pi^3 \equiv \pi^3\rho + 3a\pi^2 \\ & \cdots\cdots\cdots\cdots \\ a_n & \rho\pi^n \equiv \pi^n\rho + na\pi^{n-1} \end{array} \tag{1}$$

Adding,

$$\rho \cdot \phi(\pi) \equiv \phi(\pi) \cdot \rho + a\phi'(\pi) \tag{5}$$

(6) With equation (5),

$$\begin{aligned} \rho \cdot \phi(\pi) &\equiv \phi(\pi) \cdot \rho + a\phi'(\pi) \\ \rho^2 \cdot \phi(\pi) &\equiv [\rho \cdot \phi(\pi)] \cdot \rho + a\rho \cdot \phi'(\pi) \\ &\equiv [\phi(\pi) \cdot \rho + a\phi'(\pi)]\rho + a[\phi'(\pi)\rho + a\phi''(\pi)] \\ &\equiv \phi(\pi) \cdot \rho^2 + a\phi'(\pi) \cdot 2\rho + a^2\phi''(\pi) \\ \rho^3 \cdot \phi(\pi) &\equiv [\rho\phi(\pi)]\rho^2 + a[\rho \cdot \phi'(\pi)]2\rho + a^2[\rho\phi''(\pi)] \\ &\equiv [\phi(\pi) \cdot \rho + a\phi'(\pi)]\rho^2 \\ &\qquad + a[\phi'(\pi) \cdot \rho + a\phi''(\pi)]2\rho \\ &\qquad + a^2[\phi''(\pi)\rho + a\phi'''(\pi)] \\ &\equiv \phi(\pi)\rho^3 + a\phi'(\pi) \cdot 3\rho^2 + a^2\phi''(\pi)3\rho + a^3\phi'''(\pi) \\ &\equiv \phi(\pi) \cdot \rho^3 + a\phi'(\pi) \cdot 3\rho^2 + \frac{a^2}{2!}\phi''(\pi) \cdot 6\rho + \frac{a^3}{3!}\phi'''(\pi) \cdot 6 \end{aligned}$$

Similarly,

$$\begin{aligned} \rho^4 \cdot \phi(\pi) \equiv \phi(\pi) \cdot \rho^4 &+ a\phi'(\pi)4\rho^3 \\ &+ \frac{a^2}{2!}\phi''(\pi)12\rho^2 \\ &+ \frac{a^3}{3!}\phi'''(\pi)24\rho \\ &+ \frac{a^4}{4!}\phi^{iv}(\pi)24 \end{aligned}$$

And

$$\rho^n \cdot \phi(\pi) \equiv \phi(\pi) \cdot \rho^n + \frac{a}{1!}\phi'(\pi)n\rho^{n-1}$$
$$+ \frac{a^2}{2!}\phi''(\pi)n(n-1)\rho^{n-2}$$
$$+ \frac{a^3}{3!}\phi'''(\pi)n(n-1)(n-2)\rho^{n-3}$$
$$+ \frac{a^n}{n!}\phi^{(n)}(\pi)n! \tag{6}$$

Now, form the function

$$\psi(\rho) \equiv a_0 + a_1\rho + a_2\rho^2 + \cdots$$

as well as

$$\psi(\rho) \cdot \phi(\pi) \equiv a_0\phi(\pi) + a_1\rho\phi(\pi) + a_2\rho^2\phi(\pi) + \cdots$$

Substitute in the latter from the preceding equations, and obtain

$$\psi(\rho) \cdot \phi(\pi) \equiv \phi(\pi) \cdot \psi(\rho) + a\phi'(\pi) \cdot \psi'(\rho)$$
$$+ \frac{a^2}{2!}\phi''(\pi) \cdot \psi''(\rho)$$
$$+$$
$$+ \frac{a^n}{n!}\phi^{(n)}(\pi) \cdot \psi^{(n)}(\rho) \tag{7}$$
$$+ \cdots *$$

(7) Two correlative theorems can be formed from Eqs. (5) and (7) by the use of Theorem (3), *viz.*,

$$\pi \cdot \psi(\rho) \equiv \psi(\rho) \cdot \pi - a\psi'(\rho) \tag{8}$$

$$\phi(\pi) \cdot \psi(\rho) \equiv \psi(\rho) \cdot \phi(\pi) - a\psi'(\rho) \cdot \phi'(\pi)$$
$$- \frac{a^2}{2!}\psi''(\rho) \cdot \phi''(\pi)$$
$$- \frac{a^n}{n!}\psi^{(n)}(\rho) \cdot \phi^{(n)}(\pi) \tag{9}$$

* For the study of noncommutative operators, this theorem is a very powerful one analogous to Taylor's and Leibnitz's theorems.

(8) If $\rho \equiv D$ and $\pi = x$, Eq. (7) becomes

$$\begin{aligned}\psi(D)\cdot\phi(x) \equiv \phi(x)\cdot\psi(D) &+ a\phi'(x)\cdot\psi'(D)\\ &+ \frac{a^2}{2!}\phi''(x)\cdot\psi''(D)\\ &+ \cdots\cdots\cdots\\ &+ \frac{a^n}{n!}\phi^{(n)}(x)\cdot\psi^{(n)}(D)^* \qquad (10)\\ &+ \cdots\cdots\cdots\end{aligned}$$

This theorem is an extension of Leibnitz's theorem from which has been abstracted one of the functions, *i.e.*, $(a = 1)$.

$$\begin{aligned}\psi(D)\cdot\phi(x)\cdot S(x) = \phi(x)[\psi(D)\cdot S(x)] &+ \phi'(x)[\psi'(D)\cdot S(x)]\\ &+ \frac{1}{2!}\phi''(x)[\psi''(D)\cdot S(x)]\\ &+ \cdots\cdots\cdots\\ &+ \frac{1}{n!}\phi^{(n)}(x)[\psi^n(D)\cdot S(x)]\\ &+ \cdots\cdots\cdots\end{aligned}$$

(9) Using Eq. (1) with $\rho \equiv D$ and $\pi \equiv x$, we have $x$ and $D$ combining according to the equation

$$D\cdot x \equiv x\cdot D + a \qquad (11)$$

Directly from (11) we can obtain (10) and its correlative.

(10) Another set of results of far-reaching importance can be obtained as follows: In Eq. (5), set $\pi \equiv e^{\phi(\pi)}$ and $a \equiv 1$, obtaining

$$\rho\cdot e^{\phi(\pi)} \equiv e^{\phi(\pi)}\cdot\rho + e^{\phi(\pi)}\cdot\phi'(\pi)$$

or

$$\equiv e^{\phi(\pi)}[\rho + \phi'(\pi)]$$

giving

$$\rho + \phi'(\pi) \equiv e^{-\phi(\pi)}\cdot\rho\cdot e^{\phi(\pi)}$$

* Equation (10) was obtained in 1848 by C. J. Hargreave by the use of Leibnitz's theorem; *Phil. Trans. London* (1848), 31–54. It can be written in the symbolical form.

$$\psi(D)\cdot\phi(x) \equiv e^{D\cdot\Delta}\cdot\phi(x)\cdot\psi(D)$$

$$\text{where } D \equiv \frac{d}{dx} \quad\text{and}\quad \Delta \equiv \frac{d}{dD} \qquad (10a)$$

from which by induction we have

$$f[\rho + \phi'(\pi)] \equiv e^{-\phi(\pi)} \cdot f(\rho) \cdot e^{\phi(\pi)} \tag{12}$$

The correlative theorem is

$$f[\pi + \phi'(\rho)] \equiv e^{\phi(\rho)} \cdot f(\pi) \cdot e^{-\phi(\rho)} \tag{13}$$

(11) In Eq. (13), set $\pi \equiv x$ and $\rho \equiv D$, obtaining

$$f[x + \phi'(D)] \equiv e^{\phi(D)} \cdot f(x) \cdot e^{-\phi(D)} \tag{14}$$

the correlative being

$$f[D + \phi'(x)] \equiv e^{-\phi(x)} \cdot f(D) \cdot e^{\phi(x)} \tag{15}$$

These last two equations are remarkable extensions of Taylor's theorem. They will be found useful in the interpretation of symbolical expressions met with in the solution of differential equations.

(12) By using operators $\frac{d}{d\pi}$ and $\frac{d}{d\rho}$, Graves developed from Eq. (1) the following very general theorem which also applies to the study of differential equations. The method of induction only was used to derive it.

$$f\left[\pi_1 + \frac{d\phi}{d\rho_1}, \quad \pi_2 + \frac{d\phi}{d\rho_2}\right] \equiv e^{\phi(\rho_1,\rho_2)} \cdot f(\pi_1, \pi_2) \cdot e^{-\phi(\rho_1,\rho_2)} \tag{16}$$

(13) It is evident that when Charles Graves gave up his professorship of mathematics for the bishopric, mathematics lost a powerful generalizing type of mind. This paper was his last.

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